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On Mathematical Analyses of Critical Point Statistical Mechanics and Continuum Field Theory

by

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Abstract

Mathematical theory of critical phenomena in classical spin system is reviewed in complete detail. The main interest is to derive macroscopic critical behaviors from microscopic theories. A brief summary is given in Chapter 2.

Contents

Abstract

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Chapter 1 Introductions

Phase transitions, and critical phenomena accompanying them are no doubt extremely fascinating and amazing behaviors observed in our nature. Many excellent theoretical and experimental physicists have devoted their efforts to the investigation of phenomena. And untill now, they have brought us a lot of beautiful and successful phenomenological theories. Among them are, the mean field theory, the scaling theory, and the renormalization group theory. Looking at the brilliant successes of these theories, we might conclude that the phenomenological theories of equilibrium critical phenomena have now come to their ripening period.

On the other hand, mathematical theories of critical phenomena have also been developed. The works can be devided into two stages. The first stage is to formulate physical problems into a suitable language of mathematics. (This corresponds to stating "axioms".) As for the equilibrium statistical mechanics, this task, originated by G. W. Gibbs, is now almost completed and we have a satisfactory formalism (called Gibbsian ensemble formalism or merely thermodynamic formalism).

The second stage of mathematical theories of critical phenomena is to investigate the properties of the formalism in general or specific situations. (This corresponds to proving "theorems".) Though we have many splendid works towards the direction, the present achievement is still far from complete, when compared with the successes of such phenomenological theories as the scaling and renormalization group theories. Only few of the many conjectures proposed by the phenomenological theories were rigorously proved, and the most of the rests are still far from provable. We would like to conclude that mathematical theories of critical phenomena are still in their developing stage.

In recent years, through energetic studies in this field, it has become clear that there are rich mathematical structures in theories of critical phenomena. These structures themselves are of quite interest, and are worth studying as independent mathematical subjects.

But, a more amazing and exciting fact, which has become clearer in these years, is that quantum field theories can be also described by the same mathematical structure as that of the critical phenomena. In other words, from a mathematical point of view, these two theories are just two different faces of a single structure. This seems to be one of the most marvelous "accidental coincidences" in the nature, that man has ever experienced. The discovery of this fact brought considerable developments to quantum field theories (in particular to constructive field theories), and mathematical theories of critical phenomena.

The present thesis is a review article of these developments in the field.

In the Chapters 3 to 6. We are going to review some mathematical theories of critical phenomena, in the simplest case of classical lattice spin systems. There, we are mostly interested in how the macroscopic critical behavior can be proved from the microscopic definitions (only from which are seen no apparent indications of critical phenomena). All the individual results stated in these Chapters were (at least) once published somewhere in their original forms. But, so far as the present author knows, no review containing all these items at once was published. Here, one can view everything in a perfect consistency.

Chapter 3 contains a definition of the system. We describe, in full detail, how we deal with the equilibrium statistical mechanics of a system with infinite degrees of freedom. Correlation inequalities, which are the most useful tools in our analysis, are also discussed in this chapter.

Chapter 4 deals with modern theoretical methods to describe spin systems, which have deep connections with the quantum field theories. They were developed after the discovery of the "accidental coincidence" previously mentioned. A notion called reflection positivity plays an essential role.

After these two preliminary chapters, Chapter 5 is devoted to the study of the long-range behavior of the system in high- and low-temperature regions. Then the existence of phase transition will be established.

Chapter 6 is the heart of the present thesis. We establish the existence of the critical point, and prove some rigorous critical exponent inequalities which characterize critical phenomena.

Throughout these four Chapters, every statement is presented in a complete mathematical rigour. (Some of the well-known and technical proofs are, however, omitted except some appropriate references.)

Chapter 2 is prepared for the convenience of the readers. It contains a brief summary of the main four Chapters, a logical diagram of the notions, and a list of symbols.

The final Chapter 7 deals with some open problems in critical phenomena and continuum quantum field theories.

Chapter 2 Outline

2.1 Summary of Chapters 3 to 6

The present section contains a brief summary of the main part of the present thesis. Here, a reader can see most of the main results stated in the Chapters 3 to 6, without dealing with technical details and the complicated proofs. Moreover, if he is interested in a peculiar topic treated in the thesis, he can study the corresponding part of the article in detail, after reading this section.

First, we define our classical spin system. [Sections 3.1-2] We have a d-dimensional ($d \geqq 3$) hyper-cubic lattice Z^d, and the spin variable ϕ_x on every site $x \in Z^d$. The Hamiltonian of the system is given by.

$$H = -J \sum \phi_x \phi_y$$

where the sum is over $x, y \in Z^d$ with $|x-y| = 1$, and $J \geqq 0$. The a priori measure $d\nu(\phi)$ for the spin variables is either of the form;

$$d\nu(\phi) = (1/n+1) \sum_{j=0}^{n} \delta(-n+2j+\phi)$$

or

$$d\nu(\phi) = \text{const.} \exp(-a_1\phi^2 - a_2\phi^4 - a_3\phi^6 - \cdots) d\phi$$

The equilibrium states of the system is described by the thermal expectation formally written as,

$$< \ldots > = \text{normalization} \int_x \Pi d\nu(\phi_x) \, (\ldots) \, e^{-H}$$

where (. . .) stands for arbitrary physical quantity (observable). The thermal expectation is rigorously defined through the infinite volume limit.

Correlation inequalities are useful tools in the rigorous analysis of the spin system. [Section 3.3] We mainly make use of, the Griffiths inequalities;

$$\langle \phi^A \rangle \geqq 0, \langle \phi^A ; \phi^B \rangle \geqq 0$$

where A, B are index sets (see eq. (3.2.1)).
the Lebowitz inequality;

$$<\phi_x\phi_y\phi_z\phi_w> - <\phi_x\phi_y><\phi_z\phi_w> - \text{two permutations} \leqq 0$$

The field theoretical methods yield many useful results which are not so familiar to statistical physicists. From the property called reflection positivity [Section 4.1], we can prove, infrared bounds [Section 4.3]

$$\hat{G}(k) = \sum_x e^{ikx} < \phi_0\phi_x > \leqq \text{const.} / \mid k \mid^2 \qquad \text{if} \quad k \neq 0$$

and spectral representation [Section 4.4]

$$G(x) = < \phi_0\phi_x > = \int d\rho(\lambda, q)\, \lambda^{|x_1|}\, e^{iqx}$$

These two are the lattice theoretic versions of the Umezawa-Kamefuchi-Kälen-Lehman representation in the continuum field theory.

We apply these notions to analyze the critical behavior of the system carried by the two-point function $G(x)$.

First [Sections 5.1-2], in the high-temperature region (characterized by sufficiently small J), Simon-Lieb's correlation inequality offers us an exponentially decaying upper bound,

$G(x) \leqq$ const. $a^{|x|}$ for x sufficiently large. $a < 1$

If we combine this with the spectral representation, we obtain a sophisticated result,

$G(x) \leqq e^{-m \max(x)}$ for all x

with

$m = \xi^{-1} \doteq \lim - \ln G(x)/|x|$ (the limit exists!)

where m is the mass gap and ξ is the correlation length.

Next [Section 5.3], in the low-temperature region (characterized by sufficiently large J), the infrared bounds imply the non-clustering behavior of the two-point function.

$G(x) \to p > 0$ as $x \to \infty$

This is a consequence of the symmetry breaking.

Finally [Section 6.1], between the two regions, there exists a critical point. We establish its existence by proving the non-analytic behavior of the susceptibility,

$$\chi = \sum_x G(x) \to \infty \quad \text{as} \quad J \to J_C - 0$$

The critical phenomena observed near the critical point are characterized by the critical exponents.

$$\chi \sim (J_C - J)^{-\gamma}, \quad \xi \sim (J_C - J)^{-\nu}$$

$$G(x) \sim 1/|x|^{d-2+\eta}, \quad \text{at } J = J_C$$

We combine various methods, and prove the following rigorous inequalities for the exponents. [Sections 6.3-5]

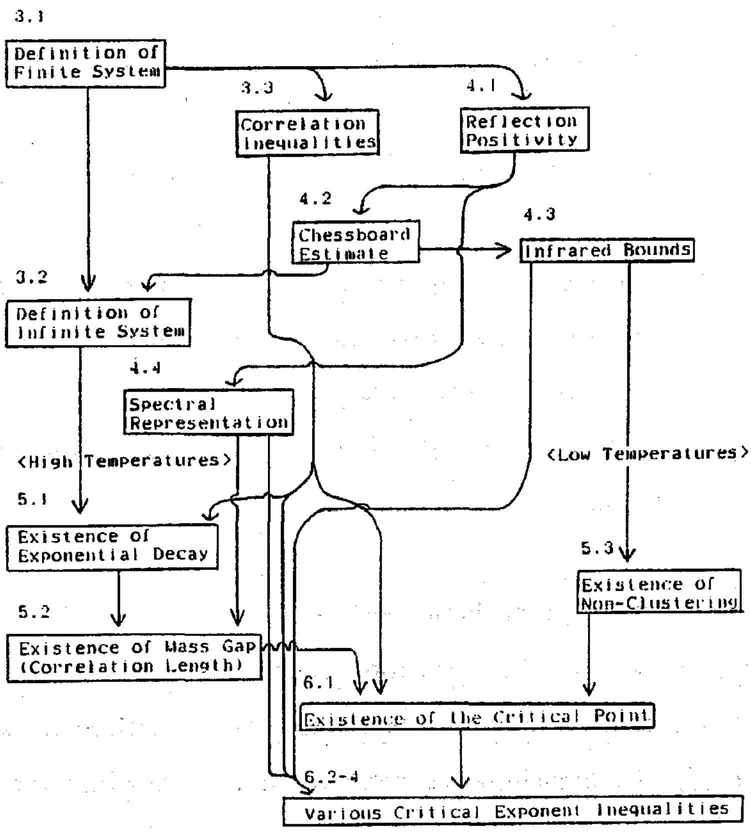
$$\gamma \geqq 1, \quad \nu \geqq 1/2, \quad 1 \geqq \eta \geqq 0$$

and

$$\nu \geqq \gamma/(2-\eta) \quad \text{(Fisher's inequality)}$$

2.2 Diagram of the Notions

The following is a logical diagram of the main notions appear in Chapters 3 to 6. (The numbers indicate the Sections.)



2.3 List of Symbols

The following is a list of the symbols appear in Chapters 3 to 6. The numbers in brackets denote the corresponding equation numbers.

A, B : index sets (3.2.1)

A, B, ... : elements of $\mathcal{A}_+$

$\mathcal{A}_L$, $\mathcal{A}_+$, $\mathcal{A}_-$: space of observables (3.1.3)

B : high temperature region (6.1.5)

$\mathcal{C}_L$: configuration space (3.1.2)

$d\nu$: a priori measure (3.1.4-6)

$d\rho$: measure in the spectral representation (4.4.14)

F : element of $\mathcal{A}$

$f(J, h)$: specific free energy (A.10)

$G(x)$: shorthand for $<\phi_0\phi_x>$

$\hat{G}(k)$: Fourier transform of $G(x)$

H : Hamiltonian (3.1.9)

$\mathcal{H}$: Hilbert space (4.4.4)

J, J_{xy} : coupling constants (3.1.9)

J_C : critical point (6.1.6)

L : finite torus-shaped lattice (3.1.1)

$|L|$: cardinality of L

L_0, L_+, L_- : sublattices (4.1.5-12)

m : mass gap (5.2.1), (5.2.4)

$M_S(J)$: spontaneous magnetization (A. 11)

$\max(x)$: maximum of $(x_1, x_2, \ldots, x_d)$

$\mathcal{N}$: null space (4.4.3)

R : set of real numbers

$r(x)$, $r_i(x)$: map on a lattice (4.1.5-12)

$t(x)$: translation operator on $\mathcal{A}$ (4.4.6)

$T(x)$: transfer matrix (operator on $\mathcal{H}$) (4.4.8)

V : finite subset of Z^d (not torus)

x, y : sites in a lattice

x_i : i-th component of x

x : $(x_2, \ldots, x_d)$

Z^d : d-dimensional hyper-cubic lattice

γ : critical exponent (6.2.5)

η : critical exponent (6.3.10)

θ, θ_i ($i=1, 2$) : reflection morphisms (4.1.1), (4.1.5-12)

$\kappa(J)$: inverse susceptibility (6.1.1)

ν : critical exponent (6.2.5)

ν_ψ : critical exponent (6.4.4)

ξ : correlation length (5.2.1), (5.2.4)

ξ_ψ : generalized correlation length of order ψ (6.4.1)

π : canonical map (4.4.5) ; or 3.141592....

ϕ_x : spin variables (3.1.2)

ϕ^A : product of spin variables (3.2.1)

χ : susceptibility (6.1.1)

ψ : order of correlation length (6.4.1)

ψ_C : critical value of ψ (6.4.7)

$\langle \ldots \rangle^H_L$: thermal expectations in a finite lattice L, with periodic boundary condition (3.1.10)

$\langle \ldots \rangle_V$: thermal expectation in a finite volume V, with free boundary condition (5.1.5)

$\langle \ldots \rangle$: thermal expectation in Z^d (3.2.5)
or shorthand of the aboves

$\langle \ldots \rangle_p$: thermal expectation in Z^d with periodic boundary condition

$\langle \ldots \rangle_f$: thermal expectation in Z^d with free boudary condition

Chapter 3 Definitions, Correlation Inequalities

3.1 Definition of a Finite System

In the present section, we give a formal definition of a spin system in a finite lattice. [R1, 116, 94] The definition only uses an integration in a finite dimensional real space. So there are no mathematical difficulties. A system in an infinite lattice (which is rather subtle) will be discussed in the next section.

First of all, we fix some general notations. Let L be a rectangular subset (see eq. (3. 1. 1)) of Z^d (d-dimensional hyper cubic lattice with $d \geqq 3$), with each pair of sites at the opposite boundaries is identified to make L a "descrete torus". (This choice of L corresponds to a periodic boundary condition. If we use a sub-lattice V without identifications of the boundary sites, it amounts to a free boundary condition.) We call L a lattice.

The elements of L are called sites and denoted by $x, y, \cdots$. To each site x, we associate a spin variable $\phi_x \in R$. The set of all possible values of ϕ_x's forms a $|L|$-dimensional ($|L|$ is the cardinality of L) eucledian space $\mathcal{C}_L$, and called a configuration space. We regard that any classical state of our system is completely described by a point of this configuration space. Then the space (or algebra) of observables $\mathcal{A}_L$ is defined as a set of all the polynomials of ϕ_x's with real coefficients.

i.e. $2\phi_x$, $\phi_x \phi_y / 2 - \phi_z^2 \cdots\cdots \in \mathcal{A}_L$

We restate these definitions before proceeding.

Definition 3.1.1:

$$L = \{(x_1 \cdots\cdots x_d) \in Z^d \mid \quad |x_i| \leqq n_i, \quad i = 1 \cdots\cdots d\} \tag{3.1.1}$$

with $d \geqq 3$, and $(x_1, x_2 \cdots\cdots n_i \cdots\cdots x_d) \sim (x_1 \cdots\cdots - n_i \cdots\cdots x_d)$

$$\mathcal{C}_L = (\phi_x \mid \phi_x \in R, \ x \in L) \tag{3.1.2}$$

$$\mathcal{A}_L = \{\text{real polynomials of } \phi_x\text{'s}\} \tag{3.1.3}$$

Remark:

The requirement we have imposed on the dimensionality of the lattice; $d \geqq 3$ is not essential (and not nescessary) in the Chapter 3 and 4. It will become quite important after section 5.3.

Next, to characterize the behavior of the bare spin variables (which corresponds to a situation of no interactions), we introduce a measure $d\nu(\phi)$ on R. The measure $d\nu(\phi)$ is called a priori measure (or a single site distribution), and defined as the following. [93, 172, 53, 52, 173]

Definition 3.1.2:

$d\nu(\phi)$ is a product measure on R, which is of the forms;

i) spin-$n/2$ Ising model;

$$d\nu(\phi) = (1/n+1) \sum_{j=0}^{n} \delta(-n + 2j + \phi)\, d\phi \tag{3.1.4}$$

ii) ϕ^4 - like (unbounded) spin systems;

$$d\nu(\phi) = \text{const} \times e^{-V(\phi)} d\phi \tag{3.1.5}$$

$$\text{where } V(\phi) = \sum_{i=1}^{M} a_i \phi^{2i} \tag{3.1.6}$$

with $M \geqq 2$, a_1 real, $a_i \geqq 0$ for $i \geqq 2$, and $a_M \neq 0$. The constant factor is chosen to make $\int d\nu(\phi) = 1$.

Remarks:

1. As a priori measure $d\nu(\phi)$, we can also take any well defined limit of the type-ii) measure. [53] An example is spin-∞ Ising model; $d\nu(\phi)=d\phi$ for $|\phi|\leqq 1$, $d\nu(\phi)=0$ for $|\phi|>1$ (3.1.7)

2. At this stage, we can equip $\mathcal{A}_L$ with a norm defined as,

$$\| F \|_0 = \int \Pi \, d\nu(\phi_x) F^2 \tag{3.1.8}$$

So, it is (mathematically) quite natural to take a completion of $\mathcal{A}_L$ with respect to this norm, and consider $\bar{\mathcal{A}}_L$ (bar denotes the completion) as a basic object [BS]. But it is not our purpose to investigate the algebraic structure of the theory, we limit ourselves to consider only the finite polynomials (i.e. the elements of $\mathcal{A}_L$).

Now, we are going to develop the equilibrium statistical mechanics of our system. Accordingly, we introduce a Hamiltonian function to characterize the interaction of the system. Suppose that we have $H\in\mathcal{A}_L$, of the form;

$$H = -\sum_{x,y\in L} J_{xy}\phi_x\phi_y \tag{3.1.9}$$

where each pairs are counted once. Couplings (J_{xy}) is determined as; $J_{xy} = J$ for $|x-y|=1$, $J_{xy} = 0$ for $|x-y|\neq 1$, with $0\leqq J<\infty$. Note that this type of interaction (characterized by positive J_{xy}'s) works to align the interacting spins to the same direction. We call such an interaction a ferromagnetic interaction, since the statistical system with these interaction is believed to being a good model of ferromagnetic materials. Moreover, our interaction is called nearest neighbour (coupling) interaction since J_{xy} is nonzero only for the nearest x, y's.

Then, we define a real valued linear function (functional) $\langle \ldots \rangle_L^H$ on $\mathcal{A}_L$ as follows.

Definition 3.1.3:

For any $F\in\mathcal{A}_L$,

$$\langle F\rangle_L^H = \int \prod_x d\nu(\phi_x) F e^{-H} \Big/ \int \prod_x d\nu(\phi_x) e^{-H} \tag{3.1.10}$$

The defining equation (3. 1. 8) only contains integrations in $R^{|L|}$. Since the convergence of the integrals is assured by the definition of $d\nu(\phi)$ and $\mathcal{A}_L$ (note that $d\nu(\phi)$ satisfies $\int d\nu(\phi)e^{a\phi}<\infty$ for arbitrary real a), $\langle F\rangle^H$ is a well defined quantity. We call $\langle F\rangle_L^H$ a thermal expectation value of the quantity F.

3.2 Infinite System

Now we discuss a definition of a system with infinite number of spin variables. The goal is an analogue of the Definition 3. 1. 3 for $L = Z^d$. However, a formal substitution of an infinite L

to the eq. (3. 1. 9) is meaningless. We must seek for a way of defining a sensible infinite volume limit.

There are several possibilities in defining infinite systems. Perhaps, the most satisfactory ways are to;

i) define a probability measure on the infinite configuration space R^∞ [Si], or

ii) define the infinite volume state on the algebra of observables for an infinite system [BR. R1. 113. 48].

These can be realized by means of some sophisticated techniques of mathematics (and mathematical physics).

Here, we do not worry about the infinite measures or states, and take the simplest approach to the infinite system. We restrict ourselves to a mere procedure of obtaining a thermal expectation value (corresponding to the infinite system) for an arbitrary polynomial of finite number of ϕ_x's.

Let L be a torus-shaped finite lattice (3. 1. 1). An index set A on L is a set of nonnegative integers suffixed by the elements of L.

$$A = \{a_x \mid x \in L\ , \quad a_x = 0, 1, 2, \cdots\cdots\} \tag{3.2.1}$$

and define $|A| = \Sigma a_x$. and supp $A = \{x \mid a_x \neq 0, \quad x \in L\}$. We write $\phi^A = \Pi \phi_x{}^{a_x}$.

Next, consider an infinite sequence of sublattices of Z^d $(d \geqq 3)$, $\{L_i\}_{i=1,2\ \ldots\ldots}$. We suppose that each lattice is again a torus of the form of eq. (3. 1. 1), and satisfies,

i) $L_i \subset L_{i+1}$, for any i (3. 2. 2)

ii) For arbitrary finite sublattice $L \subset Z^d$, there exists i such that $L \subset L_i$ (3. 2. 3)

Take an index set A with $|A| < \infty$ and supp $A \subset L_i$ for some i. (For arbitrary index set A with $|A| < \infty$, it is always possible to take such "i" thanks to the property ii).) Fix a value of the coupling constant J and, consider the expectation value $< \phi^A >^H_{L_i}$ for every finite sublattice L_i in the sequence. Then, we obtain an infinite sequence of real numbers. Important property of this sequence is characterized by the following lemma.

Lemma 3. 2. 1:

For arbitrary finite lattice L, and an index set A, with $|A| < \infty$, we have,

$$0 \leqq < \phi^A >^H_L \leqq b(A, J) \tag{3.2.4}$$

where $b(A, J)$ is a finite (and positive) constant which depends only on A and J.

The first inequality in eq. (3. 2. 4) is a consequence of Griffiths I inequality, which is discussed in the next section. As for the system with a priori measure with compact support (i. e. spins are bounded), proof of the second inequality is trivial. (We can take $b(A) = b^{|A|}$ if supp $\nu = [-b, b]$.) For a system with unbounded spins, we need a notion of reflection positivity to prove the inequality.

So we postpone the proof to Chapter 4.

Now, by the lemma, the infinite sequence of real numbers $\{<\phi^A>^H_{L_i}\}_{i=1,2,...}$ with fixed J, is confined in a compact region of R. Elementary theorem of convergence [解I] assures that we can pick up from this sequence (at least one) subsequence, which is convergent.

The same compactness argument also works, as long as we are dealing with finite types of index sets and fixed J. But what we want to consider the infinite number of index sets, and continuously varying values of J. (Suppose we are considering two-point functions. Corresponding index sets consists of all possible combinations $\{x, y\}$ of two sites x, y in the infinite lattice. The number of the combinations is of course infinite.) In such a case, we have to make use of more sophisticated compactness argument to establish the existence of a convergent subsequence.

Theorem 3.2.2:

Let $\{L_i\}_{i=1,2,...}$ be any sequence of finite lattices satisfying the conditions (3. 2. 1) and (3. 2. 2). Then there exists (at least one) subsequence

$$\{L'_i\}_{i=1,2,\cdots} \subset \{L_i\}_{i=1,2,\cdots}$$

such that $\lim_i <\phi^A>^H_{L'_i}$ exists for arbitrary index set A with $|A|$ finite, and for all values of $J \in P$. Here, P is a countable dense subset of R. (See the following proof for the definition of P.)

Proof:

Write $E_i(J, A) = <\phi^A>^{H(J)}_L$. Take an increasing sequence of positive integers $\{N_k\}_{k=1,2,...}$, with $N_k \to \infty$ as $k \to \infty$, and let

$$I_k = \{A \mid A \text{ is index set with supp } A \subset L_k, \text{ and } |A| \leqq N_k\}$$

Define finite subsets of R by

$$S_k = \{0, 1/N_k, 2/N_k, 3/N_k, \ldots, (N_k^2 - 1)/N_k, N_k\}$$

and,

$$P_k = S_1 \cup S_2 \cup \ldots \cup S_k$$

Finally, P denotes a subset of R obtained by letting k to infinity in P_k. Note that P is a countable dense subset of R. Now, fix k and consider a set of i - sequences,

$$\{E_i(J, A) \mid J \in P_k, A \in I_k\}_{i=1,2\cdots}$$

Since P_k and I_k are both finite sets, and I_k is bounded, Lemma 3. 2. 1 and the compactness argument assures that we can take a subseuqence $\{I_k(j)\}_{j=1,2,...} \subset Z_+$ such that

$$\mathrm{Lim}_j\, E_{I_k(j)}(J, A) \qquad \text{exists for all} \qquad J \in P_k, \ A \in I_k$$

Repeat this process varying k, requiring that,

$$\{I_n(j)\}_{j=1,2,\cdots} \subset \{I_k(j)\}_{j=1,2,\cdots} \qquad \text{for} \qquad k \leqq n$$

This is always possible from the conditions, $I_k \subset I_n$ and $P_k \subset P_n$. Define a new subsequence by $J(i) = I_i(i)$. Then the previous requirement yields,

$$\{J(i)\}_{i=k,\,k+1\,\cdots} \subset \{I_k(i)\}_{i=1,2\,\cdots}$$

which imply,

$\lim_i E_{J(i)}(J, A)$ exists for any A and $J \in P$.

So, $\{L_i'\}_{i=1,2,\ldots} = \{L_{J(i)}\}_{i=1,2,\ldots}$ is the desired subsequence of lattices.

Remark:

The proof presented here, is a modification of that of Ascoli-Arzela's theorem [解V; p162]. This theorem may be considered as a special case of Banach-Alaoglu theorem [RS1; p115].

Now, it is easy to define a system in the infinite lattice.

Definition 3.2.3:

For arbitrary index set A with $|A| < \infty$, and arbitrary value of the coupling $J \in P$, define

$$<\phi^A> = \lim_{i\to\infty} <\phi^A>_{L_i}^H \qquad (3.2.5)$$

where $\{L_i\}_{i=1,2,\ldots}$ is a sequence of finite sublattices of Z^d obtained from theorem 3.2.2. For $J \notin P$, we can take an increasing sequence $\{J_k\}_{k=1,2,\ldots}$ such that $J_k \in P$ and $J_k \to J$. Then we define,

$$<\phi^A>_J = \lim_{k\to\infty} <\phi^A>_{J_k} \qquad (3.2.6)$$

The limit in this definition always exists by Griffiths II inequality (see the next section). Finally, linearity extends the definition to arbitrary polynomials of ϕ^A's.

We call the thermal expectation $<\ldots>$ a limit Gibbs state obtained through the periodic boundary condition.

Though we have succeeded in defining an infinite system, it should be noted that there still remain two unsatisfactory points.

The first point is the lack of uniqueness. Theorem 3.2.2 only assures the existence of a convergent subsequence. Thus, there is a possibility that a different subsequence defines a different thermal expectation! (This fact will become essential when a first order phase transition takes place.) Thus, in the following, we suppose that we are working with an arbitrarily chosen specific infinite volume limit. Though we strongly believe in the uniqueness. (See the following remark.) This lack of the uniqueness is an essential defect of the arguments which rely on the compactness [82, 42; Section9(2)].

The second unsatisfactory point is in the definition of the expectation for $J \notin P$. Here, our definition seems quite artificial, since it relies on a monotonicity inequality. And again, the definition of the thermal expectation may be changed, if we replace an "increasing" sequence $\{J_k\}$ by a "decreasing" sequence.

Remark:

These difficulties are closely related to the boundary condition we took, i. e. periodic boundary condition. There are no such difficulties if we work out with free boundary condition. Consider an sequence of sublattices (without identifications of boundary sites) $\{V_i\}_{i=1,2,\dots}$ with the properties similar to eqs. (3. 2. 2) and (3. 2. 3). Then, Griffiths II inequtlity proves the monotonicity property,

$$\langle \phi^A \rangle_{V_i} \leqq \langle \phi^A \rangle_{V_j} \qquad i \leqq j$$

(see the next section), and we can apply the theorem of monotone convergence. So there appears no subsequences, and the infinite volume limit is unique.

Finally, we make a comment on the symmetries of the system. Our Hamiltonian (3. 1. 9) for a finite torus-shaped lattice L, is obviously invariant under the transformations;

i) $\phi_x \to -\phi_x$, for all $x \in L$

ii) $\phi_x \to \phi_{x+y}$, for all $x \in L$ with some fixed y

Accordingly, the corresponding thermal expectation satisfies,

i)′ $<\phi^A> = (-1)^{|A|} <\phi^A>$ (3. 2. 7)

ii)′ $<\phi^A> = <\phi^{A+y}>$ (translation invariance) (3. 2. 8)

Where $A+y = \{a_{x-y}\}$ for $A = \{a_x\}$. Since eqs. (3. 2. 6) and (3. 2. 7) are valid for arbitrary finite lattice L, just the same relations are valid for the infinite volume expectation $<\dots>$.

Note that, eq. (3. 2. 6) implies $<\phi^A> = 0$ for A with $|A|$ odd, in particular $<\phi_x> = 0$. Hence our system never shows the magnetization. (See the appendix to the Section 5. 3.)

3. 3 Correlation Inequalities

Before closing this preliminary chapter, we describe one of the most important and useful tools in the nonperturbative analysis of ferromagnetic spin systems and lattice field theories. They are a class of mathematical relations expressed by inequalities among the various thermal expectation values, and are called "correlation inequalities".

In the following, we state some of the wellknown correlation inequalities for a finite spin system, (which is somewhat more general than those of the Section 3. 1). It should be noted that the validities of these correlation inequalities does not depend on the structure of the lattice, though it seriously depends on the positivity of the interaction (ferromagnetic property), and the type of a priori measure. This is contrast to the various field theoretical methods described in chapter 4.

Let L be arbitrary finite set (lattice). We repeat every steps in the section 3. 1 to define a spin system. The only difference is the Hamiltonian. Here, the allowed Hamiltonian is;

$$H' = \sum_{x,y} - J_{xy}\phi_x\phi_y \tag{3.3.1}$$

where each pair is counted once, and $0 \leqq J_{xy} < \infty$. Note that this definition contains the definition in section 3. 1 (eq. (3. 1. 9)) as a special case.

The first set of inequalities is due to Griffiths. [92, 93, 94, 109, 76, 26] They characterize the fundamental property of ferromagnetic systems.

Theorem 3. 3. 1:

Let A and B be arbitrary index sets on L. Then the following inequalities are valid.

i) Griffiths I Inequality;

$$\langle \phi^A \rangle_L^{H'} \geqq 0 \tag{3.3.2}$$

ii) Griffiths II Inequality;

$$< \phi^A ; \phi^B >_L^{H'} = < \phi^A \phi^B >_L^{H'} - < \phi^A >_L^{H'} < \phi^B >_L^{H'} \geqq 0 \tag{3.3.3}$$

Note that the function $< \phi^A ; \phi^B >_L^{H'}$ denotes the intrinsic part of the interaction between ϕ^A and ϕ^B.

Remark:

The inequality can be stated in a more general setting where the allowed Hamiltonian is $H = \sum_A - J_A \phi^A$ where $J_A \geqq 0$ and the summation runs over all index sets on L. In particular, we are allowed to treat a system under positive external magnetic field. (See the appendix to the section 5. 3.)

The proofs of Theorems 3. 3. 1 is now quite well known and there has been published many review articles containing the proofs. [GJ, 172, 3, 94] Thus we omit them, by noting that the proofs are based only on a simple inequality;

$\int f(x)\, d\nu(x) \geqq 0$. where $f(x)$ is a polynomial of x with positive coefficients, and $d\nu(x)$ is a symmetric measure. [76]

Though these Griffiths inequalities might seem to be too simple and trivial, they are actually quite useful and yields very strong physical informations. [94, 56] Moreover a lot of complicated correlation inequalities can be derived using these inequalities.

As an application of the Griffiths II inequaltity, we prove a simple lemma which is useful in comparing two different systems. (It was already used in Definition 3. 2. 3. and Remark of Section 3. 2.)

Corollary 3. 3. 2:

Let H' be a ferromagnetic Hamiltonian of the form eq. (3. 3. 1), and let K be a ferromagnetic perturbation, i. e.

$$K=\sum_A -K_A\phi^A \quad \text{with} \quad K_A \geqq 0 \tag{3.3.4}$$

Then for arbitrary index set B,

$$<\phi^B>_L^{H'} \leqq <\phi^B>_L^{H'+K} \tag{3.3.5}$$

In particular, we see that $<\phi^A>$ is a non decreasing function of the couplings J_{xy}'s.

Proof:

Consider an Hamiltonian $H+tK$, $0\leqq t\leqq 1$. Then,

$$\mathrm{d}<\phi^B>_L^{H'+tK}/\,\mathrm{d}t=\sum_A K_A<\phi^B;\phi^A>_L^{H'+tK}\geqq 0$$

This Corollary will be used (frequently) in the remainder of the present thesis only by noting "from the Griffiths II inequality...".

Next, we state another correlation inequality we need in this paper.

Theorem 3. 3. 3: (Newman's Gaussian inequality)

Let F be a polynomial of ϕ_x's ($x\in L$) with positive coefficients. Then the following inequality is valid.

$$<\phi_x F>_L^{H'}\leqq\sum_{y\in L}<\phi_x\phi_y>_L^{H'}<\partial F/\,\partial\phi_y>_L^{H'} \tag{3.3.6}$$

This inequality was first proved by Newman for Ising models by a graphical method, [134, 135] and extended to some general models (spin-$n/2$ Ising models, ϕ^4-models) by Simon-Griffiths [93, 159] type argument. A direct proof of the inequality for a class of continuum spin models (which includes the systems appeared in Definition 3. 1. 2. ii)) was given by Brydges, Fröhlich and Spencer. [40, 39] They have shown that the inequality is a simple consequence of random-walk representation and the Griffiths II inequality (3. 3. 3). (We again omit the proof.)

Note that in a Gaussian systems, the inequality (3. 3. 6) is satisfied as an equality. (Thus we have the name Gaussian inequality.)

If we let $F=\phi_y\phi_z\phi_w$ in inequality (3. 3. 6), we obtain,

Corollary 3. 3. 4: (Lebowitz inequality) [115]

$$\begin{aligned} u_4(x,y,z,w) &= <\phi_x\phi_y\phi_z\phi_w>_L^{H'}-<\phi_x\phi_y>_L^{H'}<\phi_z\phi_w>_L^{H'} \\ &\quad -<\phi_x\phi_z>_L^{H'}<\phi_y\phi_w>_L^{H'}-<\phi_x\phi_w>_L^{H'}<\phi_y\phi_z>_L^{H'} \\ &\leqq 0 \end{aligned} \tag{3.3.7}$$

Note that the quantity u_4 (x, y, z, w) represents the intrinsic interaction between four spins.

At last, we state a very important theorem.

Theorem 3.3.5:

The inequalities (3. 3. 2), (3. 3. 3), (3. 3. 6), and (3. 3. 7) are all valid for the infinite volume expectation $<\ldots>$ constructed in the Section 3.2.

Proof:

The inequalities are valid for arbitrary finite lattice L. Then, they are also valid in the limit by the continuity.

Remark:

Though the inequality (3. 3. 5) is valid for $<\ldots>$, the inequality for the derivative which appears in the Proof is not always valid for $<\ldots>$. Since we do not know about the convergence of the derivative.

Chapter 4 Field Theoretical Methods

4.1 Basic Notion of Reflection Positivity

In this chapter, we establish various methods used in the analysis of spin systems which have deep relations with continuum field theories. [B, BS] They are chess-board estimate, infrared bounds, and spectral representation. The latter two notions can be regarded as incomplete descendants of the spectral representation in continuum field theories (so called Umezawa-Kamefuchi-Kälen-Lehman representation) [N, 123]. (See the discussion in Section 4.3 for the detail.)

All of these three are consequences of a single notion called reflection positivity. Reflection positivity itself is also a field theoretical concept. It was first introduced by Osterwalder and Schrader in the context of reconstructing a Mincovsky field theory from an eucledian field theory. [140, 141, 139, GJ, 86, 87]

But, at the same time, the concept has a lot to do with statistical mechanics of spin systems (it is closely related to the existence of self-adjoint transfer matrix, see section 4. 4). Systematic applications of the reflection positivity to statistical mechanics was developed by Fröhlich et al. [64, 65, 157, 59-62], and yielded many beautiful new results.

Here, we follow Fröhlich et al. [64], and discuss basic notions of the reflection positivity. In this (and in the next) section, the system under consideration is the finite system defined in section 3.1. We deal with a torus-shaped lattice L with $2n_1 \times 2n_2 \times \ldots \times 2n_d$ sites, and uniform nearest neighbour ferromagnetic interactions.

Suppose the lattice L is devided into two sublattices L_+ and L_-, with $L_+ \cup L_- = L$ and $L_+ \cap L_- =$

L_0. And also suppose that we have a one to one map $r(\ )$ from L_+ to L_- which leaves each point of L_0 fixed. Consider the algebras of the observables (see eq. (3. 1. 3)) for each lattices L_+ and L_-, and call them $\mathcal{A}_+$ and $\mathcal{A}_-$. Then we can define a linear morphism θ from $\mathcal{A}_+$ to $\mathcal{A}_-$ as the following.

Definition 4. 1. 1:

Let $x \in L_+$, define for $\phi_x \in \mathcal{A}_+$; (i. e. $x \in L_+$)

$$\theta(\phi_x) = \phi_{r(x)} \in \mathcal{A}_- \tag{4.1.1}$$

Then we define θ on whole $\mathcal{A}_+$ by the property of linear morphism.

Next, we consider the thermal expectation $<\ldots>_L^H$ defined in section 3. 1. (We omit the subscripts L and H in the following.) Then we can state the abstract definition of the reflection positivity.

Definition 4. 1. 2:

A thermal expectation $<\ldots>$ is reflection positive (on $\mathcal{A}_+$) with respect to the reflection θ, if;

$$\langle F\theta(F)\rangle \geqq 0 \qquad F \in \mathcal{A}_+ \tag{4.1.2}$$

Let $<\ldots>_0$ denote the thermal expectation of the uncoupled system. (It is obtained by setting H=0 in the definition 3. 1. 3.) Fröhlich et al. proposed the following sufficient condition for eq. (4. 1. 2).

Theorem 4. 1. 3:

A thermal expectation $<\ldots>$ is reflection positive if;

i) $<\ldots>_0$ is reflection positive.

and,

ii) The hamiltonian H can be written in the form;

$$-H = B + \theta(B) + \sum_i C_i\,\theta(C_i) \tag{4.1.3}$$

with B, $C_i \in \mathcal{A}_+$.

Proof:

Since $<F>=<Fe^{-H}>_0/<e^{-H}>_0$, we only have to show the positivity of $<F\theta(F)e^{-H}>_0$ for $F \in \mathcal{A}_+$. If we expand the exponential and use the property $F\theta(F)G\theta(G)=FG\theta(FG)$ and eq. (4. 1. 3), the quantity reduces to a sum of the terms $<A\theta(A)>_0$ with $A \in \mathcal{A}_+$ which are positive by the property i).

We point out that for the morphism θ defined from a one to one map $r(\ )$ on the lattice (as in Definition 4. 1. 1), the property i) of the theorem is always satisfied. It can be shown by noting;

$$< F\theta(F)>_0 = \int \prod_{x\in L} d\nu(\phi_x) F\theta(F)$$

$$= \int \prod_{x\in L_0} d\nu(\phi_x) \left[\int \prod_{x\in L_+ - L_0} d\nu(\phi_x) F\right]^2 \geqq 0 \tag{4.1.4}$$

for any $F\in \mathcal{A}_+$.

Remark:

Above discussion about the reflection positivity of $<\ldots>_0$ seriously depends on the assumption that $r(x) = x$ for $x\in L_0$. If not for the property, reflection positivity may fail.

Now we discuss two kinds of reflection positivities in our system. In both cases, one to one map on the lattice (denoted r) is defined as a reflection in a (hyper-)plane in the lattice. (This is, of course, the origin of the name "reflection positivity".)

Definition 4.1.4: (Reflection in a site plane)

We denote a point in L by $x = (x_1, x_2, \ldots, x_d)$. Then we define,

$$L_{1,+} = \{ x \mid 0 \leqq x_1 \leqq n_1 \} \tag{4.1.5}$$

$$r_1(x) = (-x_1, x_2, \cdots, x_d) \tag{4.1.6}$$

and denote corresponding linear morphism on $\mathcal{A}_{1,+}$ by θ_1.

In this case corresponding sublattices are,

$$L_{1,-} = \{ x \mid -n_1 \leqq x_1 \leqq 0\} \tag{4.1.7}$$

$$L_{1,0} = \{ x \mid x_1 = 0 \quad , \quad \text{or} \quad x_1 = n_1 (= -n_1)\} \tag{4.1.8}$$

Definition 4.1.5: (Reflection in a bond plane)

We define,

$$L_{2,+} = \{ x \mid 0 \leqq x_1 \leqq n_1 - 1\} \tag{4.1.9}$$

$$r_2(x) = (-x_1 - 1, x_2, \cdots, x_d) \tag{4.1.10}$$

and denote corresponding linear morphism on $\mathcal{A}_{2,+}$ by θ_2.

In this second case, the corresponding sublattices become,

$$L_{2,-} = \{ x \mid -n_1 \leqq x_1 \leqq -1\} \tag{4.1.11}$$

$$L_{2,0} = \{ \quad \}, \text{ (an empty set)} \tag{4.1.12}$$

In both cases, it easily follows from the theorem 4. 1. 3 that our thermal expectation is reflection positive with respect to each linear morphisms.

Remark:

In the proof of reflection positivity for the first case (reflection in a site plane), we do not need C_i terms in eq. (4. 1. 3). So the reflection positivity also holds, if we take the coupling constant J negative (antiferromagnetic interaction). This is contrast to the second case (reflection in bond plane), where the positivity of the interaction on reflecting bonds is essential in the proof.

4. 2 Chessboard Estimate

Now, we discuss chessboard estimate as a first application of reflection positivity. [64] And, with it, prove Lemma 3. 2. 1 which was used in the definition of the infinite system. Chessboard estimate is also a base of infrared bounds. (See the next section.)

First we investigate a main consequence of the abstract definition of reflection positivity (4. 1. 2). Define a bilinear form on $\mathcal{A}_+$ by $b(F, G)=\langle F\theta(G)\rangle$. Reflection positivity states that b is a positive (but not nescesarily positive-definite) bilinear form. Thus the Schwarz inequality is valid for b. i. e.

$$b(F,G) \leqq b(F,F)^{1/2}\ b(G,G)^{1/2} \tag{4.2.1}$$

Remark:

We recall [関I, RS1] that the Schwarz inequality follows only from the positivity, by observing $b(K, K) \geqq 0$
for $K=F/b(F, F)^{1/2}-G/b(G, G)^{1/2}$.

To see how the inequality (4. 2. 1) works, consider the simplest case. Let L be a lattice with two sites x and y. Hamiltonian of the system is defined as $H=-J\phi_x\phi_y$. Then the thermal expectation $\langle\ldots\rangle$ becomes reflection positive with respect to the morphism generated from a map $r(x)=y$. (Of course L_+ consists of single site x, and corresponding algebra $\mathcal{A}_+$ is a set of all polynomials of ϕ_x.) Let $f(t)$, $g(t)$ be arbitrary polynomials of t. Then from eq. (4. 2. 1), we have,

$$\begin{aligned}\langle f(\phi_x)\, g(\phi_y)\rangle &= \langle f(\phi_x)\,\theta(g(\phi_x))\rangle \\ &\leqq [\langle f(\phi_x)\,\theta(f(\phi_x))\rangle\langle g(\phi_x)\,\theta(g(\phi_x))\rangle]^{1/2} \\ &= [\langle f(\phi_x)f(\phi_y)\rangle\langle g(\phi_x)\,g(\phi_y)\rangle]^{1/2}\end{aligned} \tag{4.2.2}$$

Note that a product of different quantities in the L. H. S. changed to the products of the same quantities in the R. H. S., and total magnitude (dimension) of the quantity is unchanged (due to the square root). This type of estimte will turn out to be surprisingly powerful in some cases. (See the next section.)

Chessboard estimate is just a generalization of eq. (4. 2. 2) to a general torus-shaped lattice L.

Theorem 4. 2. 1: (Chessboard estimate)

Let $\{F_x(t)\}_{x\in L}$ be a set of arbitrary polynomials of t. (x is just a suffix.) Then the following inequality is valid.

$$<\prod_x F_x(\phi_x)> \leqq \prod_x <\prod_y F_x(\phi_y)>^{1/|L|} \tag{4.2.3}$$

We sketch basic ideas of the proof. For the detail, see [64]. First, observe that in our torus-shaped lattice the reflection (in bond plane) described in the previous section are not the only possible ones. Choices of specific plane of reflection are quite arbitrary and we can make use of all the reflections corresponding to all the planes (containing bonds) in the lattice. Various reflections yield various Schwartz inequalities (cf. eq. (4. 2. 1)). The inequality (4. 2. 3) is a consequence of repeated applications of these inequalities.

As a first application of the chessboard estimate, we prove the superstability lemma [148, 121, GJ] used in section 3. 2. (The following proof is due to T. Hara.)

Lemma 3. 2. 1:

For an index set A with $|A|<\infty$,

$$0 \leqq <\phi^A>_L^H \leqq b(A, J)$$

where $b(A, J)$ is a constant which depends only on A and J.

Proof:

First inequality is nothing but the Griffiths I inequality. To prove the second inequality, we use the chessboard estimate in the form,

$$<\phi^A> = <\prod_x \phi_x^{a_x}> \leqq \prod_x <\prod_y \phi_y^{a_x}>^{1/|L|} \tag{4.2.4}$$

We investigate the factor $<\Pi\phi_y^a>$ with $a>0$.

$$<\prod \phi_y^a> = \frac{\int \prod d\nu(\phi_y)\phi_y^a \exp(J\sum \phi_x\phi_{x'})}{\int \prod d\nu(\phi_y)\exp(J\sum\phi_x\phi_{x'})}$$

$$\leqq \frac{\int \prod d\nu(\phi_y)\, \phi_y^a \exp(dJ\phi_y^2)}{\int \prod d\nu(\phi_y) \exp(dJ\phi_y^2)} \tag{4.2.5}$$

where we used trivial inequalities,

$$-(\phi_x^2+\phi_y^2) \leqq 2\phi_x\phi_y \leqq \phi_x^2+\phi_y^2$$

Now, R.H.S. of (4.2.5) decouples to each site.

R.H.S. of (4.2.5)

$$= \{\int d\nu(\phi)\, \phi^a \exp(J\phi^2) \,/\, \int d\nu(\phi) \exp(-J\phi^2)\}^{|L|}$$

$$= \{c(a, J)\}^{|L|}$$

By the definition of a priori measure, $c(a, J)$ is a finite constant. Substituting this estimate into (4. 2. 4), we have,

$$<\phi^A> \leqq \prod_x c(a_x) = b(A, J)$$

4.3 Infrared Bounds

In the eucledian invariant continuum field theories, the spectral representation (Umezawa-Kamefuchi-Köllen-Lehman representation); [N, 123]

$$\hat{G}(K) = \int_{m^2}^{\infty} d\rho(a)\, 1/(k^2+a) + c\delta(k), \quad m^2 > 0 \tag{4.3.1}$$

is known to be valid. Here $\hat{G}(k)$ is a Fourier transformation of the two-point function (propagator), and $d\rho$ is some positive measure. The constant $m>0$ is called a mass gap of the theory.

A representation similar to eq. (4. 3. 1) is expected also for two-point functions of a spin systems on a lattice. In particular, if we replace $d\rho(a)$ by $\delta(a-m^2)da$, eq. (4. 3. 1) is nothing but the Ornstein-Zernike form [St; p100] of the two-point function;

$$\hat{G}(k) = 1/(k^2+m^2) \tag{4.3.2}$$

Of course this equality is a phenomenological one, and does not hold for the interacting non-trivial spin systems. (In fact, eq. (4.3.2) is valid only for continuum Gaussian model.)

As for rigorous theories of lattice spin systems, we still do not have eq. (4. 3. 1) or corresponding representation for two-point functions. All we have are two incomplete modifications of the representation.

The first one is infrared bounds. [66]

$$\hat{G}(k) \leqq \text{const. } k^{-2}. \text{ for } k \neq 0 \tag{4.3.3}$$

Note that this inequality carries no information about mass gap.

The second is the following spectral representation (for a lattice system). [150, 87,84, 161]

$$G(k_1, k) = c_k + \int_{\cosh m-1}^{\infty} \mathrm{d}\rho_k(a)\ 1/(1-\cos(k_1)+a) \tag{4.3.4}$$

The equality is somewhat similar to eq. (4. 3. 1), but lacks the eucledian invariance.

Thus, the power of the original representation (4. 3. 1) in continuum theories had to be devided into two in the lattice theory. This situation causes some difficulties in our analysis of lattice systems.

In this section, we concentrate on the first of two descendants of Umezawa-Kamefuchi-.. representation, i. e. infrared bounds.

Infrared bounds was first introduced by Fröhlich, Simon, and Spencer in their very important paper [66]. There, the existence of phase transitions in various spin systems were established for the first time. (We will discuss this proof in section 5. 3.) In [64], Fröhlich et al. showed that the infrared bounds are nothing but the consequence of reflection positivity (or chessboard estimate). Here, we are going to follow the proof indicated there.

Again, we consider the torus-shaped finite lattice L. Fourier transformation of the two point function $G(x) = \langle\phi_0\phi_x\rangle_L$ is, (we again write $\langle\ldots\rangle$ for $\langle\ldots\rangle_L^H$.)

$$\hat{G}(k) = (2\pi)^{-d/2} \sum_x e^{ikx} G(x) \tag{4.3.5}$$

where, $k = (k_1, k_2, \ldots, k_d)$, $kx = \sum_i k_i x_i$. And the Fourier inversion is,

$$G(x) = (2\pi)^{-d/2} \int \mathrm{d}k\ e^{-ikx}\, \hat{G}(k) \tag{4.3.6}$$

where $\int dk$ is a shorthand for the summation $(\pi^d/|L|)\sum_k$ in a dual space (descrete Brillouin zone).

A function $F(x)$ on L is said to be a function of positive type, if for any $N \geqq 1$ and arbitrary chosen N elements of L; x_i, $i=1, \ldots, N$, the NxN-matrix $A_{ij}=F(x_i-x_j)$ is positive. [RS2; p12]

Proposition 4.3.1:

Two-point function $G(x) = \langle\phi_0\phi_x\rangle$ is a function of positive type.

In particular, we have $G(k) \geqq 0$ (4. 3. 7)

Proof:

From translation invariance (eq. (3. 2. 7)), we can write, $G(x_i-x_j)=\langle\phi_{x_i}\phi_{x_j}\rangle$. Fix N, $\{x_i\}$, and let

q_i, $i = 1, \ldots, N$ be arbitrary complex numbers. Then,

$$\sum_{i,j} \bar{q}_i G(x_i - x_j) q_j = \langle \sum_{i,j} \bar{q}_i \phi_{x_i} \phi_{x_j} q_j \rangle$$
$$= \langle | \sum_i q_i \phi_{x_i} |^2 \rangle \geqq 0$$

Thus, $G(x)$ is a function of positive type. Then eq. (4. 3. 5) follows from Bochner's theorem. [RS2; p13]

Now we state the most important notion in the proof of the infrared bounds; Gaussian domination.

Definition 4. 3. 2:

Let $\{h_x\}_{x \in L}$ be a set of real numbers. Define,

$$Z(\{h_x\}) = \int \prod_x \mathrm{d}\nu'(\phi_x) \exp[-J/2 \sum_{x,y} \{(\phi_x - h_x) - (\phi_y - h_y)\}^2]$$

where $\mathrm{d}\nu'(\phi) = \exp(dj\phi^2)\mathrm{d}\nu(\phi)$ and the summation runs over the nearest neighbour x, y's and the each pair is counted once.

Proposition 4. 3. 3: (Gaussian domination)

For any choice of $\{h_x\}_{x \in L}$, we have,

$$Z(\{h_x\}) \leqq Z(0) \tag{4.3.9}$$

Note that, from the definition, $Z(0)$ is nothing but the usual partition function $Z = \int \prod \mathrm{d}\nu(\phi_x) \mathrm{e}^{-H}$.

Proof:

Write $\mathrm{d}\nu'(\phi) = f(\phi)\mathrm{d}\phi$, and let $F_x(\phi) = f(\phi + h_x)/f(\phi)$. Then,

$$Z(\{h_x\}) = Z(0) \langle \prod_x F_x(\phi_x) \rangle \leqq Z(0) \prod_x \langle \prod_y F_x(\phi_y) \rangle^{1/|L|}$$
$$= \prod_x [\int \prod_y \mathrm{d}\nu'(\phi_y + h_x) \exp(-J/2 \sum (\phi_x - \phi_y)^2)]^{1/|L|} = Z(0)$$

where we used the chessboard estimate.

Now we can state infrared bounds for the two-point function.

Theorem 4. 3. 4:

For the Fourier transformation of the two-point function (4. 3. 5), we have,

$$\hat{G}(k) \leqq 1/[2J(2\pi)^{d/2} \sum_{i=1}^{d} (1 - \cos k_i)] \tag{4.3.10}$$

for $k \neq 0$.

Proof:

We can rewrite the Gaussian domination (4.3.9) as,

$$Z^{-1} \int \Pi d\nu'(\phi_x) \exp[-J/2\sum \{(\phi_x-\phi_y)^2-2(h_x-h_y)(\phi_x-\phi_y)$$

$$+(h_x-h_y)^2\}]$$

$$=\langle \exp[J\sum(h_x-h_y)(\phi_x-\phi_y)\rangle \exp(-J/2\sum(h_x-h_y)^2 \leqq 1$$

Thus, $\langle \exp\{J\sum(h_x-h_y)(\phi_x-\phi_y)\rangle \leqq \exp(J/2\sum(h_x-h_y)^2\}$

If we replace $\{h_x\}$ by $\{th_x\}$ and expand the inequality in a power series of t, the first order vanishes, and we obtain from the second order,

$$\langle 1/2[J\sum(h_x-h_y)(\phi_x-\phi_y)]^2\rangle \leqq J/2\sum(h_x-h_y)^2 \tag{4.3.11}$$

Substituting (4.3.6) and $h_x = (2\pi)^{-d/2}\int dk\, e^{-ikx}\, \hat{h}(k)$ into (4.3.11), we have,

$$2J^2(2\pi)^{d/2}\int dk[\sum(1-\cos k_i)]^2\, |\hat{h}(k)|^2\, \hat{G}(k)$$

$$\leqq J\int dk \sum(1-\cos k_i)\, |\hat{h}(k)|^2 \tag{4.3.12}$$

We can let $h(k) = \delta(k-k_0)$, since $\{h_x\}$ are arbitrary, and obtain eq. (4.3.10) for nonzero k_0's.

It is easy to extend the result to an infinite lattice system.

Corollary 4.3.5:

Consider the infinite system defined in section 3.2. Then Fourier transformation of the two-point function,

$$\hat{G}(k)=(2\pi)^{-d/2}\sum e^{ikx}\langle\phi_0\phi_x\rangle$$

has a decomposition: $G(k) = c\delta(k) + g(k)$ (4.3.13)

with $c \geqq 0$ and

$$0 \leqq g(k) \leqq [2J(2\pi)^{d/2}\sum(1-\cos k_i)]^{-1} \tag{4.3.14}$$

Proof:

Take the infinite volume limit at eq. (4.3.11). Then everything follows in the same way.

4.4 Spectral Represetation

Now, it is time to dicuss spectral representation (for a lattice system): the second descendant of Umezawa-Kamefuchi-. . . representation. The derivation is again based on the reflection positivity.

But, here, the notin is used in a little bit different manner from the previous two applications, where the chessboard estimate played an essential roles. In this section, a positive bilinear form defined through reflection is a main ingredient of the analysis. Given a bilinear form, we adopt GNS construction technique and derive a representation of self-adjoint transfer matrix.

Discussions given here deeply relies on the languages of functional analysis [RS1,関I-III], and may seem less intuitive. But, the representation we obtain is very simple and useful. We are going to see in section 5.2, how strong the representation can be used in obtaining physical informations (mainly about decay properties).

For the case of continuum field theories, spectral representation based on the reflection posi--tivity is already common, and have been discussed in many literatures. [For example, 140, 141] As for lattice theories, there are few references containing detailed discussions. [87, 150, 161]

Here, we follow [161] and [87], and describe the technique in detail. Our goal is the spectral representation for two-point function in the real space; eq. (4.4.14), which is the origin of the previously mentioned formula (eq. (4.3.4)).

In this section, we directly consider an infinite system, for some reasons explained later (see Remark 4 at the end of the section). Our lattice is d-dimensional hyper cubic lattice Z^d ($d \geqq 3$), and the thermal expectation $<\ldots>$ is the one defined in section 3.2.

In this infinite lattice, we repeat every step in section 4.1 and consider the reflections. Then, all the notions developed in a finite systems can be easily extended to the infinite system, and the properties of reflection positivity still remains to be valid. (For some reasons, we here consider polynomials with complex coefficients. The proof of reflection positivity still works in the case.) For the reader's convenience, we repeat the definitions and main results.

Definitions and Theorem 4.4.1:

$L_+ = \{ x \mid x \in Z^d,\ 0 \leqq x_1 \}$

$\mathcal{A}$; polynomials of ϕ_x's, $x \in Z^d$ (and 1), with complex coefficients

$\mathcal{A}_+$; polynomials of ϕ_x's, $x \in L_+$ (and 1), with complex coefficients

r_1, r_2; reflections in hyper planes, $x_1 = 0$ and $x_1 = -1/2$

θ_1, θ_2; corresponding morphisms on $\mathcal{A}_+$

Then, $<A^*\theta_i(A)> \geqq 0$ for $A \in \mathcal{A}_+$, $i = 1, 2$. (4.4.1)

where, A^* is obtained by replacing all the coefficients in A by their complex conjugates.

First, we construct our Hilbert space. As was mentioned in section 4.2, we can define a positive bilinear form on $\mathcal{A}_+$ from the reflection morphism and the thermal expectation.

$$b(A, B) = \langle A^* \theta_1 (B) \rangle, \quad A, B \in \mathcal{A}_+ \tag{4.4.2}$$

Define the null space of $b(\ , \ )$ as,

$$\mathcal{N} = \{A \mid b(A, A) = 0, \ A \in \mathcal{A}_+\} \tag{4.4.3}$$

Then, we can define a Hilbert space by,

$$\mathcal{H} = \overline{\mathcal{A}_+ / \mathcal{N}} \tag{4.4.4}$$

where, $\mathcal{A}_+/\mathcal{N}$ is a quotient space (defined by identifying A, $B \in \mathcal{A}_+$ if $A-B \in \mathcal{N}$), and the bar denotes the completion with respect to a norm defined by $\| A \| = b(A, A)^{1/2}$. The canonical map from $\mathcal{A}_+$ to $\mathcal{H}$ is denoted by π. Then we can naturally equip to $\pi(\mathcal{A}_+) \subset \mathcal{H}$, a bilinear form $(\ , \ )$ as,

$$(\pi A, \pi B) = b(A, B), \quad A, B \in \mathcal{A}_+ \tag{4.4.5}$$

Since $\pi(\mathcal{A}_+)$ is dence in $\mathcal{H}$, $(\ , \ )$ extends to a positive bilinear scalar product on $\mathcal{H}$, and we have the full structure of Hilbert space.

Next, we discuss translation operators. For $x \in Z^d$, let $t(x)$ be a linear automorphism on $\mathcal{A}$, defined by;

$$t(x)[\phi(y)] = \phi(x+y) \tag{4.4.6}$$

Then we have,

Lemma 4.4.2:

$$x \in Z^d \text{ with } x \geqq 0, \quad t(x)\mathcal{N} \subset \mathcal{N} \tag{4.4.7}$$

Proof:

Let $A \in \mathcal{N}$, i.e. $\langle A\theta_1(A) \rangle = 0$. Since $t(x)A \in \mathcal{A}_+$.

$$\begin{aligned} 0 &\leqq \langle t(x)[A] \ \theta_1(t(x)[A]) \rangle = \langle t(x)[A] \ t(r_1(x))[\theta_1(A)] \rangle \\ &= \langle t(x - r_1(x))[A] \ \theta_1(A) \rangle = b(t(x - r_1(x)), A) \\ &\leqq b(t(x - r_1(x))[A], \ t(x - r_1(x))[A])^{1/2} \ b(A, A)^{1/2} = 0 \end{aligned}$$

Definition 4.4.3:

For $x \in Z^d$ with $x_1 \geqq 0$, define an operator $T(x)$ on $\pi(\mathcal{A}_+) \subset \mathcal{H}$ by,

$$T(x)(\pi A) = \pi(t(x)A), \quad A \in \mathcal{A}_+ \tag{4.4.8}$$

Thanks to the lemma 4.4.2. the R. H. S. of eq. (4.4.8) does not depend on the specific choice of a representative A. Thus an operator $T(x)$ is well defined. We call $T(x)$ a transfer matrix. Now we can state various properties of the operator $T(x)$.

Proposition 4.4.4:

For $x, y \in Z^d$ with $x_1, y_1 \geqq 0$, we have,

i) $$T(x)T(y)=T(y)T(x)=T(x+y) \tag{4.4.9}$$

ii) $$T(x)^{*}=T(-r_1(x)) \tag{4.4.10}$$

In particular, if we write $T_i = T(e_i)$, (e_i is the i-th unit vector of Z^d) T_1 is self adjoint, and T_i for $i \neq 1$ is unitary. Moreover,

iii) T_1 is a positive operator.

iv) $$|T(x)| = \sup_{A \in \mathcal{A}_+} (\pi A, T(x)\pi A)/(\pi A, \pi A) = 1 \tag{4.4.11}$$

Proofs:

i) We have $t(x)t(y)=t(y)t(x)=t(x+y)$ from eq. (4.4.6). Then eq. (4.4.9) follows from the definition.

ii) $$(\pi A, T(x)\pi B) = \langle A\theta_1(t(x)B)\rangle = \langle A\, t(r_1(x))\, \theta_1(B)\rangle$$

$$= \langle t(-r_1(x))[A]\, \theta_1(B)\rangle = (T(-r_1(x))\pi A, \pi B)$$

iii) It suffices to prove $(\pi A, T_1 \pi A) = \langle A\theta_1 (T(e_1)A)\rangle \geqq 0$ for all $A \in \mathcal{A}_+$. But we have $\langle A\theta_1 (T(e_1)A)\rangle = \langle A\theta_2(A)\rangle$. Thus the reflection positivity with respect to the reflection in the bond plane (θ_2-reflection positivity) proves the statement. (See eq. (4.4.1)).

iv) First note that $T\pi 1 = \pi 1$, so $|T| \geqq 1$. To see $|T| \leqq 1$,

$$(\pi A, T(x)\pi A) = b(A, t(x)A) \leqq b(A, A)^{1/2}\, b(t(x)A, t(x)A)^{1/2}$$
$$= b(A, A)^{1/2}\, b(A, t(2x)A)^{1/2}$$

repeating the process, we have,

$$\leqq b(A, A)^{1-2^{-n}} b(A, t(2^n x)A)^{2^{-n}}$$

But we have, $b(A, t(2^n x)A) = \langle \theta_1(A)\, t(2^n x)A \rangle$, so,

$$\leqq \langle \theta_1(A)\, \theta_1(A)\rangle^{1/2} \langle t(2^n x)A\, t(2^n x)A\rangle^{1/2} = \langle A^2 \rangle$$

(Here, we used the Schwartz inequality for the bilinear form $\langle AB\rangle$.) Now, $(\pi A, T(x)\pi A) \leqq (\pi A, \pi A)^{1-2^{-n}} \langle A^2\rangle^{2^{-n}}$. And letting $n \to \infty$, we have $|T| \leqq 1$.

At this stage, from the property iv), we can extend $T(x)$ to a bounded operator on whole Hilbert space $\mathcal{H}$.

Finally, we define a field operator $\Phi(x)$, analogous to the definition 4.4.3.

Definition 4.4.5:

For $x \in L_+$, define an operator $\Phi(x)$ on $\pi \mathcal{A}_+$ by

$$\Phi(x)(\pi A)=\pi(\phi(x)A), \quad A\in\mathcal{A}_+ \tag{4.4.12}$$

where $\phi(x)A$ in the R.H.S. is a mere product.

A dull part of our discussion is all finished. It is now easy to establish a useful representation for two-point function.

Lemma 4.4.5:

Two-point function of the spin system has the following Gell-Mann-Low formalism.

$$G(x)=\langle\phi_0\phi_x\rangle=(\Phi_0\Omega\,,\ T(x)\Phi_0\Omega) \tag{4.4.13}$$

where $\Omega=\pi 1$, and $x_1\geqq 0$.

Proof:

Very easy, R. H. S. = $\langle\phi_0\theta_1(\phi_x)\rangle=\langle\theta_1(\phi_0)\phi_x\rangle=\langle\phi_0\phi_x\rangle$

Theorem 4.4.6: (Spectral representation in real space)

Write $x = (x_1, x)$ for $x\in Z^d$. (x_1 needs not to be positive.) Then we have a representation for two-point function,

$$G(x)=\int d\rho(\lambda\,,\ q)\,\lambda^{|x_1|}\,e^{iqx} \tag{4.4.14}$$

where $\lambda\in[0, 1]$ and $q\in[-\pi,\pi)^{d-1}$, and $d\rho(\ \ ,\ \ )$ is a finite positive measure.

Proof:

We discuss for x with $x_1\geqq 0$. Then for $x_1\leqq 0$, the statement follows from the symmetry. Note that from poperty i) of proposition 4.4.4,

$$T(x)=T_1^{x_1}\,T_2^{x_2}\cdots\cdots T_d^{x_d}$$

Substituting the spectral representations [関 III; p401–421] for transfer matrices; $T_1 = \int\lambda\, dE(\lambda)$, $T_i = \int e^{iq_i}dF_i(q_i)$ into the Gell-Mann-Low formula (4.4.13), we obtain the desired representation (4.4.14), with a measure

$$d\rho(\lambda\,,\ q)=(\Phi_0\Omega,\ dE(\lambda)\Pi dF_i(q_i)\Phi_0\Phi).$$

This measure is positive, since dE and dF_i's are positive. $\int d\rho$ must be finite because it is equal to $\langle\phi_0{}^2\rangle$.

Remarks:

1. We can also write eq. (4.4.14) in a Fourier transformed form, as appears in eq. (4.3.4). Since the transformation requires us a careful treatment of partitially analytic functions and (moreover)

we do not need the Fourier version in the present thesis, we omit the detail of this subject. (See [150, 87].)

2. The second remark is concerned about the question; why did we have to use the reflections? For instance, consider the simplest bilinear form we can equip to the algebra $\mathcal{A}$. That is $b'(A, B) = \langle AB \rangle$. With this bilinear form and $\mathcal{A}$, we can repeat every steps in this section, and obtain a representation for two-point functions. But in this case, property ii) of proposition 4.4.4 is changed to ii)' $T(x)^* = T(-x)$. This implies that all the transfer matrices T_i are unitary. As we see in the next chapter, the term $\lambda^{|x_1|}$ (which comes from self-adjoint transfer matrix T_1) plays the most important role in the applications of the representation (4.4.14). Thus the representations arise from b' and $\mathcal{A}$ are by no means useful to physics!

3. There seems to be some more possibilities of constructing the representation by means of different bilinear forms. In particular, if one could repeat the process with $b_p(A, B) = \langle A\theta_3(B) \rangle$ (θ_3 arises form $r_3(x) = -x$), all the transfer matrices in the resulting representation would be self-adjoint. But in this case, the proof of the corresponding reflection positivity is still not available. (Or, reflection positivity may not hold.)

4. As was mentioned in the begining of the section, spectral representation is valid only in an infinite lattice system. The reason is, if L is finite, and L_+ is like in the Definition 4.1.4, we have for sufficiently large x, $T(x)A \notin \mathcal{A}_+$ even if $A \in \mathcal{A}_+$. This situation breaks the proof of the self-adjointness in Proposition 4.4.4, ii).

5. In the statistical mechanical litearatures, there often appears the notion of transfer matrix. [for example 187; Section 10] We have to distinguish this notion from our trnsfer matrix. Usual transfer matrix is defined as an operator on the algebra

$$\mathcal{A}_{\text{low}} = \{ \text{ polynomials of } \phi_x\text{'s, } x \text{ is a site on a low (hyper-plane) in the lattice } \}$$

and automatically self-adjoint. But, this transfer matrix is closely related to the partition function of the system, and not the correlations. In particular, there are no useful formula for two-point function as eq. (4.4.13).

Chapter 5 Decay Properties and Phase Transition

5.1 Exponential Clustering at High Temperatures

Now, we have finished rather lengthy preparative chapters, and are going to study miscellaneous physical properties of the system. From now on (and untill the end of the Chapter 6), we only consider an infinite lattice $Z^d (d \geqq 3)$ and corresponding thermal expectation $\langle \ldots \rangle$.

In this section, we investigate the behavior of the system when the coupling J is sufficiently small (i. e. high temperature region). In such a region, every thermodynamic quantities are known to be analytic in J [R1, 114, 116, 120], the correlation functions cluster (i. e. tend to zero) exponentially [68, 143, 50, 51, 97], and there exists a unique pure state (see the appendix to the Section 5.3) preserving all the symmetries of the Hamiltonian [49, 119, 120].

There are many ways of characterizing such high-temperature behaviors of the spin system. The most standard method is based on a rigorous version of high temperature expansions [R1]. Here, we use the method invented by Simon [158] (see also [124, 6, 112]), which relies on a correlation inequality. And we establish the exponential clustering property of the two-point function. i. e. $<\phi_0\phi_x>$ tends to zero faster than a quantity $e^{-m|x|}$ ($m>0$), when we let x to infinity.

Before discussing this clustering property, we state some preliminary inequalities representing the monotonicity of the two-point function. [129, 155, 102] These are the first applications of the spectral representation (4.4.14).

Theorem 5.1.2:

Write $G(x) = <\phi_0\phi_x>$, and $x = (x_1, x)$. Then we have,

$$G(x_1, x) \leqq G(x_1, 0) \tag{5.1.1}$$

$$G(x_1, x) \leqq G(x_1', x), \qquad x_1 / x_1' \geqq 1 \tag{5.1.2}$$

Proof:

Using the spectral representation (4.4.14),

$$G(x_1, x) = \int d\rho(\lambda, q)\, \lambda^{|x_1|}\, e^{iqx} \leqq \int d\rho(\lambda, q)\, \lambda^{|x_1|} = G(x_1, 0)$$

$$G(x_1, x) = \int d\rho(\lambda, q)\, \lambda^{|x_1|}\, e^{iqx} \leqq \int d\rho(\lambda, q)\, \lambda^{|x_1'|}\, e^{iqx}$$

$$= G(x_1', x)$$

where we used the positivity of the measure, $|e^{iqx}| = 1$ and $\lambda \leqq 1$.

Considering the symmetry of axises, it is easy to state:

$$G(x) \leqq G(\max(x)) \tag{5.1.3}$$

$$G(x) \leqq G(y), \quad \text{if} \quad x_i / y_i \geqq 1 \quad \text{for} \quad i = 1, \cdots\cdots, d \tag{5.1.4}$$

where $\max(x) = \max(x_1 \ldots x_d)$.

The main ingredient of the proof of clustering property is the following inequality due to Simon and Lieb. [158, 124]

Proposition 5.1.2:

Let V be a finite subset of Z^d containing the origin 0. Take $x \in Z^d$ outside of V. Then we have,

$$\langle \phi_0 \phi_x \rangle \leqq J \sum_{\substack{y \in V,\, y' \notin V \\ |y - y'| = 1}} \langle \phi_0 \phi_y \rangle_V \langle \phi_{y'}, \phi_x \rangle \tag{5.1.5}$$

where $<\ldots>_V$ denotes the thermal expectation (with free boundary condition) of a finite spin system consisting of sites in V.

Proof: [40, 39, 41]

We give a proof in a finite lattice ($V \subset$)L. Then the inequality is also valid in the infinite system by the convergence property. (c.f. Theorem 3.3.5.) Let H' be a Hamiltonian obtained by eliminating the terms $J\phi_y\phi_{y'}$ with $y \in V$, $y' \notin V$ (and $|y-y'|=1$) from the original Hamiltonian (3. 1. 9), and denote the corresponding thermal expectation by $<\ldots>'$. Define $F=e^{J\Sigma\phi_y\phi_{y'}}$ where the summation runs over $y \in V$, $y' \notin V$ and $|y-y'|=1$. Then it is easy to see, $<\ldots> = <\ldots F>'/<F>'$. Now we apply the Gaussian inequality (3. 3. 6) to $<\ldots>'$.

$$\langle \phi_0\phi_x \rangle = \langle \phi_0\phi_x F \rangle' / \langle F \rangle' \leqq \Sigma \langle \phi_0\phi_y \rangle' J \langle \phi_{y'}\, \phi_x F \rangle' / \langle F \rangle'$$

$$= J\Sigma \langle \phi_0\phi_y \rangle_V \langle \phi_{y'}\, \phi_x \rangle$$

Here we used the fact $<\phi^A>' = <\phi^A>_V$ if supp $A \subset V$, and $<\phi^A>' = 0$ if supp $A \not\subset V$.

Now it is easy to explain Simon's proof of the exponential clustering.

Theorem 5. 1. 3:

For sufficiently small (but nonzero) J, there exists a constant $0<a<1$ (depending on J), such that,

$$G(x_1,\ 0) \leqq \text{const.}\ a^{|x_1|} \tag{5.1.6}$$

holds for sufficiently large $|x_1|$.

Proof:

Let $A(J,\ V) = J\sum_{y \in \partial V} <\phi_0\phi_y>_V$, where ∂V denotes the boundary of V. Take V to be a sphere centered at 0, and radius D. Then from the monotonicity inequality (5. 1. 4) and the Simon-Lieb inequality,

$$G(x_1,\ 0) \leqq A(J, V)\ G(x_1 - D,\ 0) \tag{5.1.7}$$

Now for fixed V (i.e. D), $<\phi_0\phi_y>_V$ is a monotone decreasing function of J (by Griffiths II inequality (3. 3. 3)). Thus we can make the quantity $A(J,\ V)$ arbitrary small (in particular smaller than one) by letting J sufficiently small. Using (5. 1. 7) recursively, we obtain,

$$G(nD,\ 0) \leqq A^n\,\text{const.} \qquad n = 1,\ 2, \cdots\cdots \tag{5.1.8}$$

which (with the monotonicity) implies eq. (5. 1. 6).

Note that Theorem 5. 1. 3 with eq. (5. 1. 3) implies,

$$G(x) \leqq \text{const}\ a^{\max(x)}, \text{ for sufficiently large } x. \tag{5.1.9}$$

Remarks:

1. The method used here becomes much stronger when we consider geometrical informations. (See the Theorem 6. 3. 6.)

2. Of course, it is possible to make a quantitative estimate for "sufficiently small" J, which yields an upper bound for the critical temperature. [158, 149, 168, see also 175, 178]

3. For the general (not two-point) correlation functions, the clustering property also holds in the form [114, 116, 121, 58],

$$\langle \phi^A \phi^{B+x} \rangle \leqq \text{const.}\ a^{\max(x)}$$

5.2 Detailed Study of the Decay Property

In phenomenological theories of critical phenomena, the two-point function is expected to have the following scaling form (near the critical point); [St, 167, 108, 183-187]

$$G(x) = \langle \phi_0 \phi_x \rangle \sim |x|^{-(d-2-\eta)} e^{-m|x|} \tag{5.2.1}$$

where $m>0$ is an inverse-correlation length ($m^{-1}=\xi$) or a mass gap, and η is some constant (See the Section 6. 3). (Remark: In the present thesis, we always mean mass gap by m. Please do not confound mass gap m with a magnetization.)

As eq. (5. 2. 1) suggests, it is natural to define a mass gap for the real two-point function (which is not always of the form (5. 2. 1)) as follows.

$$m = \lim_{x_1 \to \infty} - \ln G(x_1, 0) / \ln x_1 \tag{5.2.2}$$

This definition is only a formal one, since we do not know whether the limit exists or not.

Now, fix the coupling J to a value in which theorem 5. 1. 3 is valid. (And we do so, throughout this section.) Then, the exponentially decaying upper bound for the two-point function assures us the existence of a lower mass gap;

$$\underline{m} = \lim_{x_1 \to \infty} \inf - \ln G(x_1, 0) / \ln x_1 \tag{5.2.3}$$

(To prove the existence of $\underline{m}$, assume it does not exist. Then it contradicts with theorem 5. 1. 3.)

Then, we combine this decay property with the spectral representation (4. 4. 14); $G(x)= \int d\rho(\lambda, q) \lambda^{|x_1|} e^{iqx}$

Proposition 5.2.1:

If $\underline{m}$ defined in eq. (5. 2. 3) exists, the measure $d\rho(\lambda, q)$ is supported on $[0, e^{-\underline{m}}] \times [-\pi, \pi)^{d-1}$.

Proof:

Write $d\rho'(\lambda) = \int_q d\rho(\lambda, q)$.

First assume that $\operatorname{supp} \rho' = [0, e^{-(\underline{m}+c)}] \subsetneqq [0, e^{-\underline{m}}]$ where $c>0$. Then,

$$G(x_1, 0) = \int_0^{\exp[-(\underline{m}+c)]} d\rho'(\lambda)\, \lambda^{|x_1|} \leqq \int d\rho'(\lambda)\, e^{-(\underline{m}+c)|x_1|}$$

$$= G(0)\, e^{-(\underline{m}+c)|x_1|}$$

This implies $\lim\inf -\ln G(x_1, 0)/\ln x_1 \geqq \underline{m}+c$ which is a contradiction to (5.2.3). Next assume $[0, e^{-\underline{m}}] \subsetneqq \mathrm{supp}\rho' = [0, e^{-(\underline{m}-c)}]$ with $c>0$. Then,

$$G(x_1, 0) = \int_0^{\exp[-(\underline{m}-c/2)]} d\rho'(\lambda)\lambda^{|x_1|} + \int_{\exp[-(\underline{m}-c/2)]}^{\exp[-(\underline{m}-c)]} d\rho'(\lambda)\lambda^{|x_1|}$$

$\geqq$ const $e^{-(\underline{m}-c/2)|x_1|}$ which again contradicts with (5.2.3).

Now it is easy to see that the lower mass gap was indeed a (real) mass gap.

Theorem 5.2.2:

If the lower mass gap $\underline{m}$ defined in eq. (5.2.3) exists, the mass gap m exists and equal to $\underline{m}$. i.e.

$$\underline{m} = m = \xi^{-1} = \lim_{x_1\to\infty} -\ln G(x_1, 0)/\ln x_1 \qquad (5.2.4)$$

Proof:

Define $\overline{m} = \lim_{x_1\to\infty}\sup -\ln G(x_1, 0)/\ln x_1$ and assume that $\overline{m} > \underline{m}$. Let $\epsilon = (\overline{m}-\underline{m})/2$. Then there exists infinite x's with $e^{-\overline{m}x_1} \leqq G(x_1, 0) \leqq e^{-(\overline{m}-\epsilon)x_1}$ which implies $\mathrm{supp}\rho' \subset [0, e^{-(m-\epsilon)}]$. This contradicts with the Proposition 5.2.1.

Thus, in this section, we started from a crude upper bound and finally proved the existence of a mass gap (=inverse correlation length). If we did not have the spectral representation, this kind of proof might become incredibly difficult, since we have to deal with both upper bounds and lower bounds.

5.3 Lack of Clustering at Low Temperatures

In contrast with the exponential clustering in the high temperature region, the two-point function does not cluster when the coupling J is sufficiently large (low temperature region). It now tends to a positive finite constant as x goes to infinity. This phenomenon is one of the consequences of symmetry breaking (see the appendix to the present Section), and is closely related to the phase transition.

In [66], Fröhlich, Simon, and Spencer showed that the existence of such a non-clustering phenomenon can be easily proven by using the infrared bounds.

Theorem 5.3.1:

Let $p = \lim_{x\to\infty} \langle\phi_0\phi_x\rangle$. Then for sufficiently large (but finite) J, we have $p>0$.

Proof:

Recall the decomposition $G(k) = c\delta(k) + g(k)$ in Corollary 4.3.5. Then we have,

$$p = (2\pi)^{-d/2} \lim_{x\to\infty} \int d^dk\, e^{\pm ikx} G(k) = (2\pi)^{-d/2} c$$

by Rieman-Lebesgue lemma [RS2].

Now, we use the infrared bounds in the form,

$$\langle \phi_0^2 \rangle = (2\pi)^{-d/2} \int d^d k \, G(k)$$
$$\leqq p + (2J)^{-1} (2\pi)^{-d} \int d^d k \, [\Sigma (1 - \cos k_i)]^{-1}$$

The integral in the R.H.S. is convergent for $d \geqq 3$. (Note that the integrand behaves as k^{-2} for small k.) If we denote its value by $I(d)$, the inequality can be written as,

$$p \geqq \langle \phi_0^2 \rangle - I(d) / 2J (2\pi)^d \tag{5.3.1}$$

Since $\langle \phi_0^2 \rangle$ is an increasing function of J, we can make the R.H.S. of (5. 3. 1) strictly positive by letting J sufficiently large.

Appendix:

In this section, we have proved that in the low temperature region, the two-point function shows the nonclustering property,

$$\langle \phi_0 \phi_x \rangle \to p > 0 \quad \text{as} \quad x \to \infty \tag{A.1}$$

At the same time, as was mentioned in the section 3. 2, we have,

$$\langle \phi_x \rangle = 0 \qquad x \in Z^d \tag{A.2}$$

which means the magnetization of the system is equal to zero. For a reader familiar with the notion of the symmetry breaking, this observation might seem strange. The purpose of this appendix is to explain how to recover the usual picture of the symmetry breaking from our theory.

A thermal expectation $\langle \ldots \rangle_1$ is said to be a pure phase (or ergodic state, or extremal state) [R1, GJ, 60, Si, BR], if it satisfies,

$$\langle \phi_0 ; \phi_x \rangle_1 = \langle \phi_0 \phi_x \rangle_1 - \langle \phi_0 \rangle_1 \langle \phi_x \rangle_1 \to 0 \quad \text{as} \quad x \to \infty \tag{A.3}$$

Our thermal expectation $\langle \ldots \rangle$ is obviously not pure from eqs. (A. 1) and (A. 2). This is where the confusion comes from.

Any mixed phase (i.e. non pure phase) can be decomposed into a linear combination of suitable pure phases as,

$$\langle \cdots\cdots \rangle = \Sigma a_i \langle \cdots\cdots \rangle_i \tag{A.4}$$

where each pure phase $\langle \ldots \rangle_i$ can be realized as a limit Gibbs state with a boundary condition B_i.

Now, from the definitions, we can write,

$$\langle \phi_0 \phi_x \rangle = \Sigma \, a_i \langle \phi_0 ; \phi_x \rangle_i + a_i \langle \phi_0 \rangle_i^2 \tag{A.5}$$

(where we assumed the translation invariance of $\langle \ldots \rangle_i$.) Then eqs. (A. 1) and (A. 3) imply,

$$\langle \phi_0 \rangle_i = p' \neq 0, \quad \text{for some} \quad i \tag{A.6}$$

which indicates the existence of the symmetry breaking.

Remarks:

1. In the low temperature region, it is expected that our thermal expectation can be decomposed as,

$$\langle \cdots\cdots \rangle = 1/2\,(\langle \cdots\cdots \rangle_+ + \langle \cdots\cdots \rangle_-) \tag{A.7}$$

where $<\ldots>_+$, $<\ldots>_-$ are the limit Gibbs states obtained through plus and minus boundary conditions respectivly [16]. In this case, eq. (A. 6) becomes,

$$\langle \phi_0 \rangle_+ = p^{1/2},\ \langle \phi_0 \rangle_- = -p^{1/2} \tag{A.8}$$

As for the two dimensional Ising model, eqs. (A. 7) and (A. 8) are true for all values of J greater than J_C [1, 103-105, 8].

2. The existence of the symmetry breaking is usually proved in a more direct way by Peierls argument [R1, Si, 145, 64, 65, 176].

We can proceed further to discuss about the notion of the spontaneous magnetization of the system. [94] Consider a Hamiltonian with external magnetic field h,

$$H(J, h) = -J\sum \phi_x\phi_y - h\sum \phi_x \tag{A.9}$$

We denote the thermal expectation obtained from $H(J, h)$ and a boundary condition B_i by $<\ldots>_{i,(J,h)}$. The specific free energy of the system is defined as,

$$f(J, h) = \lim_{L\to Z^d} 1/|L|\ \ln \int \Pi\, d\nu(\phi_x)\, e^{-H(J,h)} \tag{A.10}$$

It is known [R1] that the specific free energy does not depend on the boundary conditions.

The spontaneous magnetization of the system can be now defined as,

$$M_S(J) = \lim_{h\to+0} \partial f(J, h)/\partial h \tag{A.11}$$

Now fix a boundary condition B_i with $p'>0$ (see eq. (A. 6)). For a finite L, we have,

$$f(J, h; L, B_i) - f(J, 0; B_i) = \int_0^h dh'\, \partial f(J, h'; L, B_i)/\partial h$$

$$= \int_0^h dh'(1/|L|)(\sum \langle \phi_x \rangle^L_{i,(J,h')}) \geqq h/|L|\ \sum \langle \phi \rangle^L_{i,(J,0)}$$

where we used Griffiths inequalities. If we let $L\to Z^d$ in the inequality and substitute eq. (A. 6), we have,

$$f(J, h) - f(J, 0) \geqq p'h \tag{A.12}$$

which reduces to, $M_S(J) \geqq p' > 0$ (A. 13)

if we let $h \to 0$.

Chapter 6 Analysis of critical behavior

6.1 Existence of a Critical Point

In the last chapter, we studied some behaviors of our system in the two limitting regions characterized by small and large values of the coupling J (i.e. inverse temperature). We observed that the long-range behavior of the two-point function is quite different between the two regions. Then, it is very natural to suspect that there exists some "critical points" separating these two regions.

In fact, in the spin system we are concerning, it is generally believed that there exists a single critical point, and various thermodynamic quantities exhibit non-analyticities only at this point. [St, 167, 187] (i.e. they are analytic in any other points. [116, 122, 170, 125]) These non-analyticities are often characterized by "critical exponents", which usually play the leading roles in the theory of phase transitions. (See the Sections 6.2 to 6.4.)

Here, in this section, we state that there exixts at least one such critical point, by proving the non-analytic behavior of a certain thermodynamic function. Since the non-analyticities can take place only in an infinite system, the analysis requires us very subtle mathematical treatments of various functions. We use some correlation inequalities discussed in the previous Chapters, and try to investigate the connection between the two different regions of the coupling constants.

The thermodynamic quantity we investigate in the present section is,

$$\kappa(J) = \lim_{V \to Z^d} \left[\sum_{x \in V} < \phi_0 \phi_x > \right]^{-1} \tag{6.1.1}$$

(Note that the quantity $\sum_{x \in V} < \phi_0 \phi_x >$ is monotone increasing in V. Hence the limit always exists.) This quantity is of course the inverse of the susceptibility $\chi = \sum_x < \phi_0 \phi_x >$, but it takes only the finite values. The important character of the function $\kappa(J)$, established in the Chapter 5 is,

Proposition 6.1.1:

For sufficiently small J, $0 < \kappa(J) < \infty$

For sufficiently large J, $\kappa(J) = 0$.

Proof:

$\kappa(J) < \infty$ is a consequence of the definition. $0 < \kappa(J)$ (which is equivalent to $\chi < \infty$) follows from the exponentially decaying upper bound for the two-point function, i.e. theorem 5.1.3. Similary, non-clustering theorem 5.3.1 implies $\sum_{x \in V} < \phi_0 \phi_x > \to \infty$ as $V \to Z^d$, and $\kappa(J) = 0$.

Next, to investigate the behavior of $\kappa(J)$, we introduce the corresponding quantity for finite torus-shaped lattice L.

$$\kappa_L(J)=(\sum_{x\in L}<\phi_0\phi_x>_L)^{-1} \tag{6.1.2}$$

where $<\ldots>_L$ is the finite volume thermal expectation with periodic boundary condition. The following lemma (which, at the first glance, seems trivial) is very subtle and important.

Lemma 6.1.2:

Consider the increasing sequence of finite torus-shaped lattice used in the definition of the infinite system (Def. 3.2.3). Then we have for arbitrary value of J.

$$\lim_{L\to Z^d}\kappa_L(J)=\kappa(J) \tag{6.1.3}$$

Proof:

For $V\subset L$, (L denotes a torus-shaped lattice, and V dentoes a subregion with free boundary condition.) difine $S_{L,V}=\Sigma_V<\phi_0\phi_x>_L$. ($\Sigma_V$ stands for the summation in $x\in V$.) Let $S_1(=\kappa(J)^{-1}=\chi(J))=\lim_V\lim_L S_{L,V}=\lim_V\Sigma_V<\phi_0\phi_x>$ ($\lim_V$ is a shorthand for lim; $V\to Z^d$, and so on), and $S_2=\lim_L S_{L,L}$. We want to state $S_1=S_2$. First, fix the coupling J to a value in which $S_1<\infty$. Recall Simon-Lieb inequality in the form of eq. (5.1.7),

$$G(x_1,\ 0)\leqq A(J,\ V)G(x_1-D,0), \quad \text{with } A(J,V)=J\sum_{y\in\partial V}<\phi_0\phi_y>_V$$

Noting that $<\phi_0\phi_y>_V\leqq<\phi_0\phi_y>$ (by Griffiths II inequality), $S_1<\infty$ implies that we can let $A(J,V)<1$ for sufficiently large V. It is remarkable that the Simon-Lieb inequality is also valied for the finite volume expectation (with arbitrary boundary condition) with the same factor(!) $A(J,V)$. In particular, we have,

$$<\phi_0\phi_{x1,0}>_L\leqq A(J,V)<\phi_0\phi_{x1-D,0}>_L$$

for any L (larger than V). This, with the superstability bound (Lemma 3.2.1), implies the existence of an exponentially decaying upper bound uniform in L.

$$<\phi_0\phi_x>_L\leqq c_1a^{\max(x)}\text{for } |x|\geqq c_2$$

for sufficiently large L, with $a<1$, c_1, and c_2 independent of L. This uniform bound yields the following estimate.

i) For any $\epsilon>0$, there exists V_1 such that; for any L, V with $V_1\subset V\subset L$, $|S_{L,V}-S_{L,L}|<\epsilon/3$

The remainder is easy. Since $<\phi_0\phi_x>_L$ converges to $<\phi_0\phi_x>$, also does the finite sum, and;

ii) For any $\epsilon>0$, and V, there exists L_1, such that; for any L with $L_1\subset L$, $|S_{Zd,V}-S_{L,V}|<\epsilon/3$

Finally, since we have assumed $\lim_V S_{Zd,V}=S_1<\infty$, thus,

iii) For any $\epsilon>0$, there exists V_2 such that; for any V with $V_2\subset V$, $|S_1-S_{Zd,V}|<\epsilon/3$

i)-iii) together imply,

*) For any $\epsilon>0$, there exists L_3 such that; for any L contains L_3, $|S_1-S_{L,L}|<\epsilon$ which means $S_2=\lim_L S_{L,L}=S_1$.

Next, consider the case $S_1=\infty$. Then instead of the property 1), we can make use of the tirvial relation;

i)' $S_{L,V}\leqq S_{L,L}$.

Then the analogous discussion implies $S_2=\infty$.

Remark:

If one deals with the thermal expectation with free boundary condition, the proof of the corresponding proposition;

$$\lim_V \lim_{V'} \Sigma_V <\phi_0\,\phi_x>_{V'} = \lim_V \Sigma_V <\phi_0\,\phi_x>_V$$

becomes remarkably simpler, since $\Sigma_V<\phi_0\phi_x>_{V'}$ is increasing in both V and V'. But in the case, the foregoing proofs will become complicated. [42;p27–34] We are going to deal with the free boundary condition in the Section 6.3.

The following theorem, which establishes the existence of a critical point, is the main result of the present section.

Theorem 6.1.3:

The function $\kappa(J)$ is a continous funciton of J.

Proof:

Take $J_1<J_2$. Then,

$$\begin{aligned} 0 &\leqq \kappa_L(J_1)-\kappa_L(J_2) = -\int_{J_1}^{J_2} dJ\, d\kappa_L(J)/dJ \\ &= -\int_{J_1}^{J_2} dJ\,(-\kappa_L(J)^2\,\Sigma<\phi_0\phi_x\,;\,\phi_y\phi_{y'}>_L) \\ &\leqq \int_{J_1}^{J_2} dJ\,\kappa_L(J)^2\, 2\Sigma<\phi_0\phi_y>_L<\phi_x\phi_{y'}>_L \\ &\leqq 2\int_{J_1}^{J_2} dJ = 2(J_2-J_1) \end{aligned}$$

where we have used the Lebowitz inequality (3.3.8). Now, by Lemma 6.1.2, we can take the limit $L\to Z^d$ in the above inequality to obtain,

$$0\leqq \kappa(J_1)-\kappa(J_2)\leqq 2(J_2-J_1)\,,\quad \text{for } J_1<J_2 \tag{6.1.4}$$

which establishes the (Lipschitz) continuity of $\kappa(J)$.

Now, we define the high-temperature region and the critical point as the following.

Definition 6.1.4:

$$B=\{J|\kappa(J)\neq 0\}\subset R_+=\{J\,|\,J\geqq 0\} \tag{6.1.5}$$

$$J_c=\inf B \tag{6.1.6}$$

From the Proposition 6.1.1, we know that the high-temperature region B is not empty, and $0<J_c<\infty$. Moreover the Theorem 6.1.3 implies,

Corollary 6.1.5:

B is a connected open subset of R_+.

$$\text{i.e. } B=[0, J_c), \text{ and } \kappa(J_c)=0 \tag{6.1.7,8}$$

Now the theorem and the corollary imply,

$$\kappa(J)\to 0, \text{ as } J\to J_c-0 \tag{6.1.9}$$

or (noting that $\chi(J)=\kappa(J)^{-1}$),

$$\chi(J)\to\infty, \text{ as } J\to J_c-0 \tag{6.1.10}$$

These relations establish the existence of the critical point in the theory.

Remarks:

1. The existence of the critical point was first proved by Baker [10, 11], and Mcbryan and Rosen [128, 146]. See also the elegant description of Brydges, Fröhlich, and Sokal. [42]
2. The facts that $\kappa(J)=0$ for large J and $\kappa(J)\neq 0$ for small J does not always imply $\kappa(J)\to 0$ as $J\to J_c-0$, since the function $\kappa(J)$ can be discontinuous at J_c. Such a situation is expected for the models which undergo first-order phase transitions. [111] (See the Remark 2 after Proposition 6.3.3)

6.2 Exponents γ and ν

Now, we are going to study the non-analytic behavior of the macroscopic (thermodynamic) functions at the critical point in a further detail. In the previous section, we established the existence of the critical point through the following behavior of the function $\kappa(J)$ $(=\chi(J)^{-1})$.

$$\kappa(J)\to 0 \text{ or } \chi(J)\to\infty \text{ as } J\to J_c \tag{6.2.1}$$

Similarly, we can state the singular behavior for the mass gap (i.e. inverse correlation length) of the theory. (See the section 5.2 for the definitions.)

Theorem 6.2.1:

$$\text{We have } m(J)\to 0 \text{ or } \xi(J)\to\infty, \text{ as } J\to J_c-0 \tag{6.2.2}$$

Proof:

Recall that in the spectral representation (4.4.14), the support of the measure $\rho(\ ,\ )$ was written as $[0, e^{-m}] \times [-\pi, \pi)^{d-1}$ with the mass gap m. (Proposition 5.2.1) Then the spectral representation implies,

$$\sum_{x_1} G(x) = \int d\rho(\lambda, q) \sum_{x_1} \lambda^{|x_1|} e^{iqx} \leqq \int d\rho (1-\lambda)^{-1} e^{iqx}$$

$$\leqq (1-e^{-m})^{-1} G(0, x)$$

Using this relation recursively in each d-coordinates, we obtain,

$$\sum_{x} G(x) \leqq (1-e^{-m})^{-d}, \quad \text{which implies,}$$

$$0 \leqq (1-e^{m(J)})^{d} \leqq \kappa(J) \tag{6.2.3}$$

Letting J sufficiently near J_c, this inequality reduces to

$$0 \leqq m(J)^{d} \leqq \text{const.}\, \kappa(J), \quad \text{or} \quad \chi(J) \leqq \text{const.}\, \xi(J)^{d} \tag{6.2.4}$$

These, combined with the eq. (6.2.1) implies the present theorem.

These behavior of the functions $\kappa(J)$ and $m(J)$ motivate us to define the critical exponents.

Definition 6.2.2:

The constants γ and ν are defined as the following limits.

$$\gamma = \lim_{J \to J_c - 0} \ln \kappa(J) / \ln(J_c - J) \tag{6.2.5}$$

$$\nu = \lim_{J \to J_c - 0} \ln m(J) / \ln(J_c - J) \tag{6.2.6}$$

In the conventional notations, these definition are written as,

$$\kappa(J) \sim (J_c - J)^{\gamma} \quad \text{or} \quad \chi(J) \sim (J_c - J)^{-\gamma} \tag{6.2.7}$$

$$m(J) \sim (J_c - J)^{\nu} \quad \text{or} \quad \xi(J) \sim (J_c - J)^{-\nu} \tag{6.2.8}$$

Remarks:

1. The difinition 6.2.2 contains assumptions about the existences of the limits. At present, we still can not remove these assumptions for the general models in the consideration. Of course, it is possible to treat only the well-defined quantities (such as $\gamma_{\sup}$=lim sup $\ln\kappa/\ln(J_c-J)$, $\gamma_{\inf}$=lim inf..., etc. with ∞ allowed for their values), and develop the remainder of the theory with these quantities. But such a task is nothing but an abstract nonsence, hence we avoid it.

2. For the special models, we can prove the existence of the limits in eqs. (6.2.5) and (6.2.6). They are two-dimensional Ising model, and some models in $d>4$ dimensions. (See the Section 7.2 of the present thesis for the latter.)

It is now easy to establish some inequalities for the critical exponents.

Theorem 6.2.3:

For the critical exponents γ and ν, we have the inequalities,

$$\gamma \geqq 1 \tag{6.2.9}$$

$$\nu \geqq 1/d \tag{6.2.10}$$

Proof:

Use the inequality (6.1.4) with $J_1=J<J_c$, $J_2=J_c$. Then, we have,

$$0 \leqq \kappa(J) \leqq 2(J-J_c)$$

which implies the desired inequality (6.2.9). The inequality (6.2.10) is then a simple consequence of eq. (6.2.4).

Remark:

We are going to prove an improved lower bound for the exponent ν in the Section 6.4. (Corollary 6.4.7)

6.3 Decay Property at the Critical Point, Exponent η

This section is concerned with the structure of the system at the critical point. The main feature is a peculiar decay property of the two-point function, characterized by a power low.

Here, we encounter a very subtle and complicated problem about the boundary condition of the system. To the author's regret, we have to choose either of the following two ways before proceeding.

i) Make an additional assumption on the behavior of the order parameter at the critical point.

ii) Use a new boundary condition (free boundary condition), from now on.

If we persist in rigorous theories, the second way is more satisfactory. There, we do not have to make any assumptions, and we can enjoy all the results in the present thesis in a mathematical completeness. But, at the same time, we have to work hard to compare the two different boundary conditions. Inevitably, the discussions will become quite complicated and difficult. Moreover, such a change of boundary condition injures the beauty and the consistency of the thesis. (See note added.)

So, if the reader does not prefer tedious discussions on the boundary conditions, he is suggested to accept the following very plausible assumption and skip to the page 784. (This is the first way.)

Assumption 6.3.1:

The critical point J_c is in the single phase region. i.e. the order parameter is equal to zero at the critical point.

$$p = \lim_{x\to\infty} < \phi_0 \phi_x > = 0 \quad \text{at} \quad J = J_c \tag{6.3.1}$$

The statement in the assumption is generally believed as a fundamental character of second order phase transitions. The facts proved in the Sections 7.2 and 7.3 strongly suggest that our system undergoes second order phase transition. But they are not sufficient to prove the assumed statement.

Now, we describe how to avoid the assumption by using the thermal expectation obtained from the free boundary condition. (This is the second way.)

Remark or Execuse:

The reader might wonder why didn't we use the free boundary condition from the beginning of the present thesis. As was noted in the Remark 2 after the definition 3.2.3, the use of the free boundary condition simplifies some of our discussions. But, so far as the author knows, the proof of the infrared bounds (and the Gaussian domination) for the free boundary condition expectation is not published. And he does not know how to prove it. (Of course, infrared bounds are strongly expected to be valid for the infinite volume thermal expectation obtained from the free boundary condition. We can even find some suggestions towards the proof in [64].)

We recall that the free boundary condition thermal expectation is obtained as a limit,

$$< \ldots >_{\mathrm{f}} = \lim_{V\to Z^d} < \ldots >_V \tag{6.3.2}$$

where $<\ldots>_V$ denotes the finite volume expectation with the free boundary condition. It is obtained by replacing the torus-shaped lattice L by a sub lattice V in the Definition 3.1.3. Observe that if we cut some bonds in the torus-shaped lattice L, we obtain a free boundary condition sub lattice. Combined with the Griffiths II inequality, this fact implies,

$$< \phi^A >_V \leqq < \phi^A >_L \tag{6.3.3}$$

which, in the infinite volume limit, reduces to,

$$< \phi^A >_{\mathrm{f}} \leqq < \phi^A >_{\mathrm{p}} \tag{6.3.4}$$

Here, $<\ldots>_{\mathrm{p}}$ is the inifnite volume thermal expectation with periodic boundary condition (which had been denoted merely $<\ldots>$ in the present thesis).

Very important relation between these two thermal expectations is,

Proposition 6.3.2:

Critical points of the two thermal expectations exactly coincides. i.e.

$$\chi_{\mathrm{f}}=\Sigma_x<\phi_0\,\phi_x>_{\mathrm{f}}<\infty \quad \text{if and only if} \quad \chi_{\mathrm{p}}=\Sigma_x<\phi_0\,\phi_x>_{\mathrm{p}}<\infty \tag{6.3.5}$$

Proof:

From eq. (7.3.4), we have $\chi_{\mathrm{f}}\leqq\chi_{\mathrm{p}}$. Thus $\chi_{\mathrm{p}}<\infty$ implies $\chi_{\mathrm{f}}<\infty$. Next, assume $\chi_{\mathrm{f}}<\infty$. Then, we can take a sufficiently large region V and make the quantity $\Sigma_{x\in\partial V}<\phi_0\phi_x>$ smaller than one. Then, from the Griffiths II inequality we have $A(J,V)=\Sigma_{x\in\partial V}<\phi_0\phi_x>_V<1$. Applying the Simon-Lieb inequality to $<\ldots>_{\mathrm{p}}$ in the form $<\phi_0\phi_x>_{\mathrm{p}}\leqq A(J,V)<\phi_0\phi_{x-D}>_{\mathrm{p}}$ (c.f. eq. (5.1.7)), we conclude that $<\phi_0\phi_x>_{\mathrm{p}}$ also shows the exponential decay. Hence $\chi_{\mathrm{p}}<\infty$.

Remark:

Moreover, we strongly expect that the two thermal expectations $<\ldots>_{\mathrm{p}}$ and $<\ldots>_{\mathrm{f}}$ completely coincides for $J\leqq J_{\mathrm{c}}$. [116] But, we lack the proof. (See note added.)

From eqs. (6.3.4) and (6.3.5), we know that,

i) The two point function $<\phi_0\phi_x>_{\mathrm{f}}$ decays exponentially, if $J<J_{\mathrm{c}}$.

and,

ii) $\Sigma_x<\phi_0\phi_x>_{\mathrm{f}}=\infty$ for $J_{\mathrm{c}}\leqq J$

with finite nonzero J_{c}. These correspond to the results we obtained in the Chapter 5. As for the continuity of the inverse susceptibility established in the Section 6.1, the things does not go so easily. In fact, we have to restate the Theorem 6.1.3 for $<\ldots>_{\mathrm{f}}$, independently to the previous proof. (This is the most unsatisfactory point in the present thesis!) The proof is described in the paper of Brydges, Fröhlich, and Sokal's [42; Prop. 5.1] in a quite detail, so we here omit it.

Then, all the results in the Section 6.2 becomes valid for the free boundary condition expectation as well. Thus from now on, we use the free boundary condition thermal expectation as our main thermal expectation. (After the page 784, we denote $<\ldots>_{\mathrm{f}}$ by merely writing $<\ldots>$.)

For this expectation, we have the following proposition which yields a statement corresponding to the Assumption 7.3.1.

Proposition 6.3.3:

For $J\leqq J_{\mathrm{c}}$, we have,

$$0 \leqq \langle \phi_0 \phi_x \rangle_{\mathrm{f}} \leqq \mathrm{const.} / J\,(|x|+1)^{d-2} \tag{6.3.5}$$

Proof:

Using the infrared bounds (eqs. (4.3.13) and (4.3.14)), and the generalized Young inequality [RS2; p30–32], Sokal was able to prove,

$$\sum_{x,y \in V_r} \langle \phi_x \phi_y \rangle_{\mathrm{p}} \leqq p + \mathrm{const.} / J r^{d-2}$$

where V_r is a hypercubic region with sides of length r, and p is an order parameter. (See eq. (6.3.1)) This inequality combined with Hegerfeldt's inequality [102, 161],

$$\langle \phi_0 \phi_{(a,\,0,\,0,\ldots)} \rangle \leqq \langle \phi_0 \phi_{(y_1, y_2, \ldots, y_d)} \rangle$$

whenever $|a| \geqq \Sigma |y_i|$, yields the bound,

$$\langle \phi_0 \phi_x \rangle \leqq p + \mathrm{const.} / J\,(|x|+1)^{d-2}$$

which is known as the Sokal's real space version of the infrared bounds. [164; Lemma a.3] But, at the present, the proof of the Hegerfeldt's inequality is known only for the free boundary condition expectations. Thus, all that we can prove here is, (using eq. (7.3.4))

$$0 \leqq \langle \phi_0 \phi_x \rangle_{\mathrm{f}} \leqq p_{\mathrm{p}} + \mathrm{const.} / J\,(|x|+1)^{d-2} \tag{6.3.6}$$

where p_{p} is an order parameter of the periodic boundary condition expectation. For $J<J_{\mathrm{c}}$, we have $p_{\mathrm{p}}=0$, so the eq. (6.3.6) reduces to the desired eq. (6.3.5).

To extend this to the critical point $J=J_{\mathrm{c}}$, note that the Griffiths II inequality implies,

$$0 \leqq \langle \phi_0 \phi_x \rangle_V \leqq \langle \phi_0 \phi_x \rangle_{\mathrm{f}} \leqq \mathrm{const.} / J\,(|x|+1)^{d-2}$$

for $J<J_{\mathrm{c}}$. But finite volume expectation $\langle \phi_0 \phi_x \rangle_V$ is continuous in J, the bound for the quantity is also valid for $J=J_{\mathrm{c}}$. Taking the infinite volume limit, we obtain eq. (6.3.5).

Remarks:

1. The only technical difference between the periodic boundary condition and the free boundary condition is that the former lacks the motonicity inequality,

$$\langle \phi_0 \phi_x \rangle_V \leqq \langle \phi_0 \phi_x \rangle_{\mathrm{f}}$$

which enabled us to extend the inequality (7.3.6) to the critical point.

2. We could prove the statement corresponding to the Assumption 6.3.1 for the free boundary condition expectation. But, we have to emphasize that this statement represents the property of the free boundary condition itself, rather than the property of the system.

To make the situation clearer, consider a priori measure not of the form defined in the Section 3.1. (e.g. $d\nu(\phi) = \{(1-2a)\delta(|\phi|-1)+a\delta(\phi)\}\, d\phi$) Then, the system with certain a priori measure (e.g. the above one with sufficiently large a) is expected to undergo the first order phase transition. [GJ, M. Suzuki; private communication] In such a case, though the Gaussian, the Lebowitz, and the Simon-Lieb inequalities become invalid, the Griffiths I and II inequalities and the infrared bounds still remains to be valid. Then (see [128; appendix]), we can prove, "$<\phi_0\phi_x>_f \to 0$ at the critical point." in the similar way. This apparently seems to be contrary to the existence of the first order phase transition, but is actually not.

At the critical point (of the first order phase transition), we have a phase coexistence. There are (at least) three different pure phases at $J=J_c$. One of them is a para-phase characterized by the exponential clustering, and the other two are the symmetry broken phases. The above proof indicates that the free boundary condition picks up only the para-phase at the critical point. On the other hand, the rigorous analysis of the Potts' model due to Kotecky and Shlosman [111] strongly suggests that the periodic boundary condition expectation is a mixed phase of all the pure states possible at the critical point.

Briefly speaking, the periodic boundary condition expectation at the critical point carries the information of both the high-temperature phase and the low-temperature phase, while the free boundary condition expectation carries the information of the high-temperature phase only. Hence the proof of the Assumption 6.3.1 is easier in the free boundary condition.

3. The analysis of the low-temperature phase is incredible difficult, when compared with that of the high-temperature phase. Only few correlation inequalities are useful in this region, and almost none has been proven about the critical phenomena. (See for example [95, 89, 90].)

This is where the reader should skip to.

Now, at the critical point $J=J_c$, the two-point function $G(x)=<\phi_0\phi_x>$ satisfies.

$$G(x) \to 0 \quad \text{as} \quad x \to \infty \tag{6.3.7}$$

(by the Assumption 6.3.1 or the Proposition 6.3.3)

and,

$$\sum G(x) = \infty \quad \text{(by the Corollary 6.1.5)} \tag{6.3.8}$$

Here, $<\ldots>$ stands for the infinite volume thermal expectation with the periodic boundary condition (if the reader accepted the Assumption 6.3.1), or the free boundary condition (if he did not).

The eqs. (6.1.7) and (6.1.8) can not be satisfied if $G(x)$ is decaying exponentially. So, it is natural to expect the decay proportional to some power of the distance.

$$G(x) \sim 1/|x|^{d-2+\eta} \tag{6.3.9}$$

where η is some constant. The formal definition is,

Definition 6.3.4:

Define a cirtical exponent η by,

$$\eta + d - 2 = \lim_{x_1 \to \infty} -\ln G(x_1, 0)/\ln x \tag{6.3.10}$$

Again (see the Remark after the Definition 6.2.2), we have made a technical assumption on the existence of the limit. See also the Remark 2 after the Lemma 6.4.5, where an alternate definition of η without any assumption is described.

Note that eqs. (6.3.7) and (6.3.8) imply the trivial critical exponent inequalities,

$$2 - d \leqq \eta \leqq 2 \tag{6.3.11}$$

In the remainder of the present section, we discuss the improvements of the eq. (6.3.11). The first one is a consequence of the infrared bounds.

Theorem 6.3.5:

The following inequality is valid.

$$0 \leqq \eta \tag{6.3.12}$$

Proof:

It is a direct consequence of the Proposition 6.3.3 (if the reader is dealing with the free boundary condition). Or, we can prove this directly from the infrared bounds. (See [66].)

The next improvement is due to Simon, and is based on the sophisticated version of the geometric method appeared in the Section 5.1. [158]

Theorem 6.3.6:

If the two point function $G(x)=\langle\phi_0\phi_x\rangle$ has a bound of the form,

$$G(x) \leqq \text{const.}|x|^{-q} \tag{6.3.13}$$

with $q>d-1$, $G(x)$ inevitably exhibits the exponential decay. Thus the critical exponent η must satisfy,

$$\eta \leqq 1 \tag{6 3.14}$$

Proof:

Assume that the eq. (6.3.13) is valid. Recall the Simon-Lieb inequality in the form, $G(x_1, 0) \leqq A(J, V)G(x_1-r, 0)$ with $A(J, V)=J\Sigma_{y\in\partial V}\langle\phi_0\phi_x\rangle_V \leqq J\Sigma_{y\in\partial V}G(y)$

If we choose V to be a spherical region with radius r, we have from eq. (6.3.13),

$$A(J, V) \leqq J \text{ const} \cdot r^{d-1-q}$$

From the assumption $q>d-1$, the R.H.S. can be made smaller than one, by letting r sufficiently large. Hence $G(x)$ decays exponetially.

Note added: (April, 1984)

The cojecture given in the Remark after Propositon 6.3.2 had turned out to be provable. i.e. The thermal expectations obtained through periodic and free boundary conditions exactly coincides in the high-temperature region ($J<J_c$). This simplifies some of the complicated discussions in Section 6.3.

The explicit statement is

Theorem:

Assume ($\chi_p<\infty$ or)$\chi_f<\infty$. Then for any index set A with $|A|<\infty$, we have $<\phi^A>_p = <\phi^A>_f$.

Proof:

Consider a rectangular parallelepiped region $V\subset Z^d$. V can be made into a torus L by adding some bonds on its boundary ∂V. Consider a thermal expectation $<\ldots>_{V,\alpha}$ corresponding to a system with couplings J for bonds inside V, and αJ for those in ∂V. Then we observe that

$$<\cdots>_{V,0} = <\cdots>_V \;,\quad <\cdots>_{V,1} = <\cdots>_L$$

Now, we can compare the tow finite-volume thermal expectations as

$$0 \leqq <\phi^A>_L - <\phi^A>_V = \int_0^1 d\alpha \frac{d}{d\alpha} <\phi^A>_{V,\alpha}$$

$$\leqq \int_0^1 d\alpha\, J \sum_{(y,y')\in\partial V} <\phi^A ; \phi_y \phi_{y'}>_{V,\alpha}$$

$$\leqq J \sum_{(y,y')\in\partial V} \sum_{x\in A} <\phi_x \phi_y>_p <\phi^{A-x} \phi_{y'}>_p$$

where we used Griffiths and Gaussian inequalities. Note that, since $\chi_f<\infty$, $<\phi_x\phi_y>_p$ has an exponentially decaying upper bound. (See proof of proposition 6.3.2.) When we let $V\to Z^d$, $<\phi^A>_L$ and $<\phi^A>_V$ converge to $<\phi^A>_p$ and, $<\phi^A>_f$ respectively, and the R.H.S. of the above inequality converges to zero since the distance between ∂V and supp$A$ increases to infinity.

I am grateful to Hiroshi Watanabe for pointing me out the proof.

6.4 Fisher's Inequality

In the present section, which is the last section of the main body of the present thesis, we discuss one of the most beautiful critical exponent inequalities which relates the three critical exponent, γ, ν, and η. The inequality was first proved by Fisher [56], and thus is called Fisher's inequality. Here, we present a proof of the inequality proposed by Hara and Tasaki, which seems to be the simplest derivation of the inequality (and some other inequalities). [99, 126, 127]

When Fisher published his result in 1969, he had to make many asumptions on the critical behavior of the system to state the inequality. But, in more than ten years since his original paper, most of the assumptions were proved from the microscopic theory, (as one can see in the previous part of the present thesis). And now, we can state Fisher's inequality with (almost) no assumptions. (See the Remark 1 after the Definition 6.2.2)

Here, to avoid undesired complicated notations, we are going to deal with the correlation length ξ (instead of the mas gap m), and the susceptibility χ (instead of the inverse susceptibility κ). There will be no confusions if one keeps in mind the relations,

$$\xi = m^{-1}, \quad \chi = \kappa^{-1}$$

Now, to investigate the critical behavior of the correaltion length ξ, we introduce an intermidiate quantity ξ_ψ, which is expected mimic the behavior of ξ.

Definition 6.4.1:

For arbitrary nonzero real number ψ, the generalized correlation length of order ψ is defined as,

$$\xi_\psi = [\lim_{V\to Z^d} \sum_{x\in V} (|x_1|+1)^\psi G(x) / \sum_{x\in V} G(x)]^{1/\psi} \tag{6.4.1}$$

where $G(x) = <\phi_0 \phi_x>$.

Note that the quantity inside the limit is a monotone increasing function of V. Thus the limit is either convergent or strictly divergent. Hence the quantity ξ_ψ is well defined if we allow ∞ as its value.

The following two properties of ξ_ψ is very important.

Lemma 6.4.2:

For arbitrary (fixed) value of J, ξ_ψ is a non-decreasing function of ψ. i.e.

$$\xi_\psi \leqq \xi_{\psi'} \quad \text{for any} \quad \psi \leqq \psi' \quad \text{with} \quad \psi, \psi' \neq 0 \tag{6.4.2}$$

Proof:

Assume $0<\psi_2<\psi_1$ or $\psi_1<\psi_2<0$. Then, Hölder's inequality, [RS2;p32]

$$\left(\sum_x a(x)^p\right)^{1/p}\left(\sum_x b(x)^q\right)^{1/q} \geqq \sum_x a(x)b(x)$$

where, $a(x)$, $b(x) \geqq 0$, p, $q>1$, $p^{-1}+q^{-1}=1$

with $a(x)=(|x_1|+1)^{\psi_2}G(x)^{1/p}$, $b(x)=G(x)^{1/q}$, and $p=\psi_1/\psi_2$

implies the non-decreasing property in each cases. Next, let $\psi>0$, and use the Schwartz inequality,

$$\left(\sum_x a(x)^2\right)^{1/2}\left(\sum_x b(x)^2\right)^{1/2} \geqq \sum_x a(x)b(x)$$

with $a(x)=(|x_1|+1)^{\psi/2}G(x)^{1/2}$, $b(x)=(|x_1|+1)^{-\psi/2}G(x)^{-1/2}$

Then, we have, $\xi_{-\psi} \leqq \xi_\psi$.

These two together prove the non-decreasing property on whole R-$\{0\}$.

Lemma 6.4.3:

We have the inequality,

$$\xi_1(J) \leqq \text{const.}\ \xi(J) \tag{6.4.3}$$

provided that ξ is sufficiently large (say $\xi \geqq 1$). Here, ξ is the correlation length defined by $\xi=m^{-1}$ (See the Theorem 5.2.2).

Proof:

The spectral representation (4.4.14) implies,

$$\sum_x(|x_1|+1)G(x)=\int d\rho(\lambda,q)\sum_{x_1}(|x_1|+1)\lambda^{|x_1|}\sum_x e^{iqx}$$

$$=(2\pi)^{-(d-1)}\int d\rho(\lambda,0)[(1+\lambda)/(1-\lambda)^2+\lambda(1-\lambda)/(1-\lambda)^2]$$

$$\leqq (2\pi)^{-(d-1)}\int d\rho(\lambda,0)(1+\lambda)/(1-\lambda)^2$$

$$\sum_x G(x)=\int d\rho(\lambda,q)\sum_{x_1}\lambda^{|x_1|}\sum_x e^{iqx}$$

$$=(2\pi)^{-(d-1)}\int d\rho(\lambda,0)(1+\lambda)/(1-\lambda)$$

Thus, by noting $\lambda \in [0, e^{-m}]$ (Proposition 5.2.1),

$$\xi_1=\sum(|x_1|+1)G(x)/\sum G(x) \leqq (1-e^{-m})^{-1}=(1-e^{-1/\xi})^{-1}$$

which implies eq. (6.4.3) for sufficiently large ξ (or ξ_1). (If we require $\xi \geqq 1$, for instance, then the constant can be chosen to e=2.71828...)

Remark:

The analogue of the Lemma 6.4.3 holds for ξ_ψ with arbitrary $\psi\neq 0$. i.e. $\xi_\psi(J)\leqq \text{const}(\psi)\xi_\psi$ [164;(2.44), 162] But for our purpose, the simplest case is sufficient.

Now, we define critical exponent for ξ_ψ, analogous to eq. (6.2.6).

$$\nu_\psi = \lim_{J\to J_c-0} -\ln \xi_\psi(J)/\ln(J_c - J) \tag{6.4.4}$$

Then we have the following inequality for the critical exponents.

Proposition 6.4.4:

We have the inequalities,

$$\nu_\psi \leqq \nu_{\psi'} \quad \text{for} \quad \psi \leqq \psi', \quad \psi, \psi' \neq 0 \tag{6.4.5}$$

and

$$\nu_\psi \leqq \nu \quad \text{for} \quad 0 \neq \psi \leqq 1 \tag{6.4.6}$$

Proof:

They are the staraightforward consequences of the Lemmas 6.4.2 and 6.4.3, and the definition of the critical exponents.

Remark:

Again the inequality (6.4.6) is valid for all $\psi\neq 0$.

The central feature of our proof is that we investigate the behavior of ξ_ψ (and ν_ψ) for negative ψ's. Then, we find,

Lemma 6.4.5:

Let $$\psi_c = \eta - 2 < 0 \tag{6.4.7}$$

(c.f. eq. (6.3.8)) Then for $\psi<\psi_c$, we have an equality,

$$\nu_\psi = \gamma/|\psi| \tag{6.4.8}$$

Proof:

Let $G(x,J)$ denote the value of $\langle\phi_0\phi_x\rangle$ when the coupling is J. Fix $0<J_1<J_c$ and write $I=[J_1, J_c]$. For arbitrary $J\in I$ and $\epsilon>0$, we have,

$$G(x, J_1) \leqq G(x, J) \leqq G(x, J_c) \leqq \text{const.}/|x|^{d-2+\eta-\epsilon}$$

with a suitable constant depending only on ϵ. The first two inequalities follow from the Griffiths II inequality. The last inequality is a consequence of the definition of η (eq. (6.3.4)). Now, consider the quantity,

$$\chi_\psi(J) = \Sigma\,(|x_1|+1)^\psi G(x)$$

From the above inequality, we have for $J \in I$,

$$C_1 \leqq \chi_\psi(J) \leqq \Sigma \text{const.} (|x_1|+1)^\psi / |x|^{d-2+\eta-\varepsilon}$$

$$\leqq \text{const.} \int d^d x (|x_1|+1)^\psi / |x|^{d-2+\eta-\varepsilon}$$

where $C_1 = G(0, J_1) > 0$. The integral in the R.H.S. can be evaluated as, $\int dr\, r^{\psi+1-\eta+\epsilon}$ and is convergent if $\psi < \eta - 2 - \epsilon = \psi_c - \epsilon$. Thus, for any $\psi < \psi_c$, we put $\epsilon = (\psi_c - \psi)/2$ and use these estimate to obtain,

$$0 < C_1 \leqq \chi_\psi(J) \leqq C_2 < \infty \quad \text{for any} \quad J \in I = [J_1, J_c]$$

Noting that $\xi_\psi = [\chi(J)/\chi_\psi(J)]^{1/|\psi|}$ for $\psi < 0$, we obtain $\xi_\psi(J) \sim \text{const.}\chi(J)$ for $\psi < \psi_c$ which implies eq. (6.4.8).

Remarks:

1. The essence of the proof is in the fact that $\chi_\psi(J_c) < \infty$ for $\psi < \psi_c$. This property may seem to be relying on the definition of η (which contains the unproved assumption on the existence of the limit). But, if we use the Proposition 6.3.3, we can rigorously prove that $\chi_\psi(J_c) < \infty$ for $\psi < -2$. Then noting that $\chi_\psi(J_c)$ is strictly increasing in ψ, we observe that, $\psi_c = \sup\{\psi \mid \chi_\psi(J_c)\} < \infty$ is a well defined quantity and satisfies the inequality $-2 \leqq \psi_c \leqq -1$. (See the Section 6.3.) Such being the case, our proof does not rely on the specific difinition or the unproved assumptions.
2. Moreover, one can define η (without any assumptions) by $\eta = \psi_c + 2$ using the above ψ_c.

It is now very easy to prove Fisher's inequality.

Theorem 6.4.6:

For the critical exponents γ, ν, and η, the following inequality is valid.

$$\nu \geqq \gamma / (2 - \eta) \tag{6.4.9}$$

Proof:

Combinig eqs. (6.4.5), (6.4.6), and (6.4.8), we have,

$$\nu \geqq \nu_1 \geqq \nu_\psi = \gamma / |\psi| \quad \text{for} \quad \psi < \psi_c = \eta - 2 < 0$$

Letting $\psi \to \psi_c - 0$, we obtain eq. (6.4.9).

We point out that this inequality, combined with the inequalities $\gamma \geqq 1$ and $\eta \geqq 0$ (eqs. (6.2.9) and (6.3.12)) implies an improved lower bound for the exponent ν.

Corollary 6.4.7:

$$\nu \geqq 1/2 \tag{6.4.10}$$

Remarks:

1. The equality corresponding to eq. (6.4.9), $\nu = \gamma/(2-\eta)$ is known as the scaling relation and is

expected to hold in various spin systems. [108, St, 167] But the rigorous proof of the equality is still far beyond the reach.

2. In contrast, the equalities corresponding to eqs. (6.2.9), (6.3.6), and (6.4.10), $\gamma=1$, $\eta=0$, and $\nu=1/2$ are the results of mean-field theory, and do not hold in general cases. (c.f. Section 7.3)

3. The method explained in this section yields other critical exponent inequalities if we make some assumptions on the critical behavior. [99, 126, 127]

Chapter 7 Some Problems in Continuum Field Theories and Critical Phenomena

7.1 Continuum Field Theories and Critical Phenomena

≪ From Statistical Mechaniacs to Field Thoery ≫

Today, it has become common that continuum field theories can be defined as continuum limits of suitable lattice field theories. This procedure can be summarized as follows.

i) Choose an ϵ-dependent ($0<\epsilon\lesssim$const) Hamiltonian (or action) of lattice theory $H(\epsilon)$, so that $\epsilon\to 0$ corresponds to the critical point of the statistical mechanical system defined by,

$$< \dots >_{\varepsilon} = Z_{\varepsilon}^{-1} \int \Pi \, d\phi_x (\dots) \, e^{-H(\varepsilon)} \tag{7.1.1}$$

This choice of Hamiltonian corresponds to the renormalization of coupling constants and mass.

ii) Choose a normalization function $f(\epsilon)$ (corresponds to field strength renormalization), so that the limits in iii) exist.

iii) Define continuum Schwinger functions by,

$$S^{(n)}(x_1, \dots, x_n) = \lim_{\varepsilon \to 0} f(\varepsilon)^n < \phi_{[x_1/\varepsilon]} \cdots \phi_{[x_n/\varepsilon]} >_{\varepsilon} \tag{7.1.2}$$

where [. . .] denotes the nearest integer.

At the first glance, this procedure seems to be very difficult (or even impossible). For example, in the procedure ii), we have to assure the convergence of the Schwinger functions for every x_i's and n's, by dealing with only a single factor $f(\epsilon)$. But if we apply some techniques useu in the analysis of lattice systems, the things become surprisingly simpler. Once the convergence was established for $n=2$, then the Gaussian inequality assures the convergence for all $n\geqq 4$, (This requires an argument similar to that used in the proof of the theorem 3.2.2) The convergence for $n=2$ can be proven if we know that the correlation length ξ diverges as the critical point is approached (i.e. $\epsilon\to 0$) [82, 164, 42]

The more interesting and more difficult part of the theroy is to investigate the structure of the continuum limit. The simplest and the most basic argument towards the direction is the triviality (or the nontriviality) of the continuum limit. It can be stated by asking whether the connected four point function;

$$u_4(x_1, x_2, x_3, x_4) = S^{(4)}(x_1, \cdots, x_4) - S^{(2)}(x_1, x_2) S^{(2)}(x_3, x_4) \text{ -two permutations}$$

is equal to zero (then the limit is said to be trivial), or not (then the limit is nontrivial). If the theory is trivial, it is physically equivalent to the free field. [B, GJ, 2. 7]

Now the question is; how can we know the property of the continuum limit? Fix a Hamiltonian $H(\epsilon)$, and suppose that we know everything about the critical phenomena which takes place when we let ϵ to zero. Then, it is very easy to investigate the limit (7.1.2), and we can derive all the properties of the continuum limit from the information of the critical phenomena. In this sence, continuum field theory is contained in a theory of critical phenomena.

If we proceed on this dogma, the simplest choice of the ϵ-dependent Hamiltonian may be,

$$H(\varepsilon) = -(J_c - \varepsilon) \sum_{|x-y|=1} \phi_x \phi_y - \sum_x V(\phi_x) \tag{7.1.3}$$

where J_c denotes the critical value of the coupling corresponding to a priori measure $e^{-V(\phi)} d\phi$. Then, from analysis done in the previous chapters of the present thesis, we know that $\xi \to \infty$ as $\epsilon \to 0$. Thus we can prove the existence of (at least one) continuum limit field theory.

Then how about the triviality or nontriviality? It can be shown [1, 7] that the nontriviality of the limit corresponds to the hyper-scaling law $d\nu = 2\Delta_4 - \gamma$ among the critical exponents, if we are constructing a massive field theory. Alas, our knowledge on the critical phenomena is still too poor to prove (or disprove) the hyper scaling relation in the general system (7.1.3).

The only succesful results in this direction are the scaling limit of the tow-dimensional Ising model based on the exact solution, and the beautiful theory of triviality in $d>4$ ϕ^4-type field theories due to Aizenman and Fröhlich. [2, 3, 5, 63, 7, 101]

≪ From Field Theory to Statistical Mechanics ≫

Now let us change our point of view, and consider what can we learn about the critical phenomena from the knowledge of continuum field theories.

Briefly speaking, the merit of this program is the following. When we investigate the critical phenomena of a statistical mechanical system, a very difficult task is to look just at the critical point. The difficulty comes from the situation that we can find no apparent sign of the critical point in the basic definition of the system. [Remark: In the author's opinion, the major success of the

Kadanoff-Wilson's renormalization group theory relies on the fact that it enabled us to characterize (though in a phenomenological way) the critical point as a fixed point of some transformation.] But as was described previously, a continuum quantum field theory can be regarded as a critical point of the corresponding statistical mechanical system. Thus, it may be possible to deal only with the critical point by investigating the continuum quantum field theory.

Recently, in their very important paper, Brydges, Fröhlich, and Sokal proved the existence and the nontriviality of $\lambda\phi^4$ systems in d=2 and 3 dimensions. [41, 42, 164] Their method is, in principle, a realization of the procedure i)–iii) described at the beginning of the present section. But in their analysis, they did not deal with the critical phenomena of the system directly. Instead, they set the ϵ-dependent Hamiltonian (action) in just the form indicated by the traditional renormalized perturbation theory (up to the second order of the coupling λ). And, with the aid of a new set of very useful inequalities called skeleton inequalities, they proved that everything predicted by the renormalized perturbation theory can be made rigorous. Then all the procedures i) to iii) can be easily executed by repeating what we did in the perturbation theory. This method forms sharp contrast with the (imaginary) program of the construction we have described previously.

Now, to execute the program of ≪ from field theory to statistical mechanics ≫ in this example, we write down the ϵ-dependent Hamiltonian (action) used by Brydges-Fröhlich-Sokal.

$$S_\varepsilon = 1/2\sum\varepsilon^{d-2}(\phi_x - \phi_y)^2 + a(\varepsilon)/2\sum\varepsilon^d\phi_x^2 + \lambda/4\sum\varepsilon^d\phi_x^4 \tag{7.1.4}$$

where $a(\epsilon)$ is the bare mass function written as,

$$a(\varepsilon) = a_0 - \lambda C_1(\varepsilon)\varepsilon^{-1} + \lambda^2 C_2(\varepsilon)\ln\varepsilon \qquad (d=3)$$

$$a(\varepsilon) = a_0 - \lambda C_3(\varepsilon)\ln\varepsilon \qquad (d=2)$$

with the functions of order 1; $C_i(\epsilon)$, i=1, 2, 3. We can translate this into the statistical mechanical convention by changing the normalization of spin variables by,

$$\phi = \varepsilon^{-(d-2)/2}\varphi$$

$$S_\varepsilon = -\sum\varphi_x\varphi_y + (d + a(\varepsilon)\varepsilon^2/2)\sum\varphi_x^2 + \lambda\varepsilon^{4-d}/4\sum\varphi_x^4 \tag{7.1.5}$$

It is easily observed that when ϵ=0, the Hamiltonian (action) corresponds that of massless Gaussian model (if $d \geqq 3$). Thus, we can expect that some critical phenomena takes place when we let ϵ to zero. Moreover, from the successful theory of Brydges-Fröhlich-Sokal's, we know rigorously that this critical phenomena satisfy the hyperscaling relation in d=2 and 3 dimensions! Thus, as for the very strange statistical mechanical system (7.1.5), our program worked successfully.

≪ Universality. . . ? ≫

Does the above analysis give us any information on the critical phenomena of the "usual" statistical mechanical system? Compare the Brydges-Fröhlich-Sokal's Hamiltonian (action) with the statistical mechanical Hamiltonian obtained by substituting the ϕ^4-a priori measure into eq. (7.1.3).

$$H_\varepsilon = -(J_c - \varepsilon)\sum \phi_x \phi_y + a' \sum {\phi_x}^2 + (\lambda'/4)\sum {\phi_x}^4 \tag{7.1.6}$$

Two Hamiltonians (7.1.5) and (7.1.6) both describe the ϕ^4-model and have their critical points at ϵ=0. But two Hamiltonians look quite different. When compared with the moderate ϵ-dependence of eq. (7.1.6), the field theoretical Hamiltonian (7.1.5) changes quite radically with ϵ. Though we have some sophisticated knowledges on the critical phenomena of the system (7.1.5), it seems hopeless to extract any informations on the critical phenomena in (7.1.6).

But, if we recall the picture of fixed points and critical surfaces proposed by Wilson [187, 183, 184, 186], we can imagine that the two apparently different Hamiltonians describe the same single critical phenomena. If this is true, the field theoretical analysis of Brydges-Fröhlich-Sokal can offer a proof of the hyperscaling relation in the statistical mechanical system (7.1.6). And we can conclude a lot in both statistical mechanics and the continuum field theory.

Remarks:

1. Some crude analysis on the universality picture and the hyper scaling relation can be found in [156]. See also [13, 15].

2. There are quite many attempts towards the establishment of the rigorous version of the renormalization group theory. For examples; [Si, 9, 12, 27, 30, 31, 71–75, 96, 137, 138]

3. We find enormous literatures on the constructive field theory, other than those referred in the text. As for the statistical mechanical construction, [98] is a recent review written in Japanese. [GJ, 16, 139, 86] can be also read as review articles. [140, 141] by Osterwalder-Schrader and [131–133] by Nelson contain the basis of the constructive approach. A series of papers due to Glimm-Jaffe [77–88] are already classical. Balaban's and Battle III-Federbush's approaches [17–22, 23–25] relies on a rigorous version of the renormalization group. See also [28–29], [151–154].

3. In the construction of the ϕ^4 field theory of Brydges-Fröhlich-Sokal, the rotational invariance of the resulting continuum limit was left unproved. It seems possible to prove the rotational invariance by using a suitably defined rotationally invariant lattice such as random lattice. [43–45]

The first step is to construct a Gaussian model (lattice free field theory) on the lattice. It may be done by, first working on the continuum limit, and then considering a "descrete approximation" of the solution.

Then, we consider a $\lambda\phi^4$-perturbation to the descrete Gaussian system, and construct its continuum limit. This will be done by comparing the perturbed field and Gaussian field in just the way developed by Brydges-Fröhlich-Sokal. If we average over the random lattices, we will obtain a continuum field theory which is nontrivial and invariant under the rotation.

7.2 Triviality, Mean-Field Properties, and Universality

≪ Gaussian Properties ≫

In 1982, Aizenman and Fröhlich independently proved so called triviality of the $\phi^4{}_d$-continuum field theory in $d>4$ dimensions. [2, 3, 5, 63, 7, 101] They established that any continuum limit of the theory obtained from the lattice cutoff (and from the single-phase region) is inevitably a free field theory (i.e. a Gaussian theory). This theory can be regarded as a first rigorous version of the Wilson's interpretation of the continuum field theories as an infrared stable fixed pints in the phase space. [182, 187] Again, if the Wilson's picture is completely true, the theory of triviality must imply that the critical phenomena of the ϕ^4-model are mean-field like in $d>4$ dimensions. In fact, Aizenman was also able to prove the equality for critical exponent; $\gamma=1$ in the same paper. [4, 3, 5] But the mean-field properties of the other exponents were left unproved. (This is a manifestation of the fact that the theory of critical phenomena includes the continuum field theory, but the converse is not true. See the Section 7.1.)

As for the mean-field properties, a new tool was developed by Brydges-Fröhlich-Sokal (but they have not published this application, as far as the author knows) in their paper on the construction of $\phi^4{}_d(d=2,3)$ field theories. [41, 39, 40, 32] If we apply their skeleton inequalities to the $\lambda\phi^4$-spin systems, we can prove the critical exponent equalities:

$$\gamma=1, \quad \Delta_4=3/2, \quad \alpha=0 \tag{7.2.1}$$

provided that the coupling λ is sufficiently small and the space dimesionality is greater than four. [100, 98]

The values of the exponents appear in eqs. (7.2.1) all coincide with those predicted by the mean-field theory. Thus, we have obtained a proof for one of the conjectures of the renormalization group theory.

But, if we look at the mechanism of the proof of the equalities (7.2.1), it becomes clear that the theory has little to do with the idea of the renormalization group. Rather, it can be regarded as a rigorous version of the perturbation expansion from the mean-field theory. [37, 33, 34, 38]

The essence of the proof is in the fact that the loop diagrams appear in the expansion of four-point function are divergence free, and only the tree diagrams are relevant in $d>4$ dimensions.

How about the other mean-field properties? (such as, $\nu=1/2, \eta=0, \ldots$) It may seem that these can be solved by the similar method as the previous ones. But the perturbation expansion of the two-point function is not so clearcut as that of the four-point function. The behavior of the two-point function cannot be fully characterized by the tree diagrams only, even in the $d>4$ dimensions. It is pointed out (T. Hara, private communicaion) that we have to deal with infinite numbers of diagrams in order to completely characterize the $d>4$ dimensional systems.

≪ Universality, again ≫

According to this spirit, we can proceed to prove the universality of the ctitical phenomena in a quite restricted form. That is; the equalities (7.2.1) are also valid for the systems in $d>4$ dimensions with $V(\phi)=\Sigma\lambda_i\phi^{2i}$ (where, λ_1 real and λ_i's are positive for $i\geqq 2$. In fact, we can treat a little more generalized potential, see [100].) provided that λ_i's are sufficiently small. The proof is again based on a rigorous version of the perturbation expansion and uses the generalized skeleton inequalities proved by Hara, Hattori, and the present author.

Remark:

It is desireble that we could prove more stronger universality in general (not $d>4$) dimensional systems. That is, for example, to state;

$$\gamma=\gamma'', \quad \Delta_4=\Delta''_4, \quad \nu=\nu'', \quad \eta=\eta'', \quad \cdots \tag{7.2.2}$$

where γ, Δ_4, ν, η, ... are the critical exponents of some $\lambda\phi^4$ system, and γ'', Δ''_4, ν'', η'', ... are those of a different system. (e.q. $\lambda'\phi^4$ system with different λ',or $\lambda\phi^4+\lambda''\phi^6+\ldots$ system). It is now rather doubtful that the skeleton-type inequlity is stronger enough to prove these statements (even partially).

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2) Papers

*** The following nonstandard abbreviations are used. ***

C.M.P. = Communications in Mathematical Physics

J.M.P. = Journal of Mathematical Physics

J.S.P. = Journal of Statistical Physcis

Bielefeld 1975 = Quantum Dynamics : Models and Mathematics (Proceedings of the symposium at CIR, Bielefeld Univ. 1975), L. Streit/ed., Acta Physica Austriaca Suppl. XVI, Springer-Verlarg, Berlin-Heidelberg-N.Y., 1976

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Remark:

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